DE CALCULO

FLUENTIUM

LIBRI DUO.

Quibus Subjunguntur

LIBRI DUO

DE OPTICA ANALYTICA.

Authore

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LONDINI:

Ex Officina *Pearsoniana* MDCCXVIII.

PRÆNOBILI ET

ILLUSTRI VIRO
Vice-Comiti *Stanhope*,

DE MAHON

BARONI DE ELVASTON,

Sacræ Regiæ Majeſtati a Secretis, &c.

SCientiarum incrementa ex *Geometriâ* præſertim eſſe petenda tam evidenter conſtat, ut prolixè id oſtendere inutile videatur. Si *Philoſophiam Naturalem* ſpectemus, pauca admodum in illa magni pretii aliunde quam ex *Geometria* & Experi-

bus, deducenda inveniemus. Nec minùs conſpicuus eſt illius uſus in pleriſq; magni momenti negotiis: Maxima utiq; hominum commercia, inter Gentes longo Terræ Mariſq; tractu diſſitas, ſine Arte Navigandi geri nequeunt; Artemq; illam *Geometriæ* & *Aſtronomiæ* principiis inniti notiſſimum eſt: Sed Artes omnes a *Geometria* pendentes hic enumerare nolo; Unicam tantùm (eamq; non prætereundam) Artem ſcilicet Belli memorare lubet: In qua, Vir Nobiliſſime (ſalvâ ſumma tua modeſtia ſit dictum) te tam egregiè excellere teſtantur Bella Hiſpanica ut inter primarios hujus *Sæculi* Duces meritò annumereris. Quum itaq; pateat *Geometriæ* uſus in omnibus ferè tam Belli quam Pacis artibus, nemo mirabitur quod hunc Tractatum Mathematicum Tibi conſe-

consecraverim, qui in utrâq; Teipsum tam inclytum reddideris, ut dictu sit difficile utrum celebrior sub Regno antecedenti Dux Exercitûs extiteris, an consummatior Reipublicæ Minister sub præsenti Imperio Sapientissimi Optimiq; Regis, ex cujus æqua & stabili administratione omnia expectare licet, quæ vel ad ipsius gloriam vel ad Subditorum prosperitatem conducere possunt. Dubitare itaq; non possumus, quin Tibi (cui tam grata sunt Literariæ pariter ac Civilis Reipublicæ commoda) *Philosophiæ* etiam & Matheseos incrementum curæ non minimæ sit futurum. Ignoscas mihi interim quæso meam audaciam, qui Te magnis Regni negotiis occupatum tam diu nugis meis interpellaverim. Et ut

omnia, quæ ad utramq; Rempublicam promovendam meditaris, feliciter ſuccedant, Deum Opt. Max. precatur

Vir Nobiliſſime

Servus tuus Humillimus

Et Devotiſſimus

Jo. Craig.

PRÆFATIO AD

LECTOREM.

HAbes hic B. L. *quæ multos ante annos de Calculo fluentium sum meditatus, & cujus prima Elementa, cum Juvenis essem, circa Annum* 1685 *excogitavi: Quo tempore* Cantabrigiæ *commoratus D.* Newtonum *rogavi, ut eadem, priusquam prælo committerentur, perlegere dignaretur: Quodq; Ille pro summa sua humanitate fecit: Nec-non ut Objectiones in Schedulis meis contra D.* D. T. *allatas corroboraret, duarum Figurarum Quadraturas sponte mihi obtulit; erant autem harum Curvarum Æquationes* $m^2 y^2 = x^4 + a^2 x^2$ *&* $m y^2 = x^3 + a x^2$; *Meque interim certiorem fecit se posse hujusmodi innumeras exhibere per* Seriem Infinitam, *quæ in datis conditionibus abrumpens Figuræ propositæ Quadraturam Geometricam determinaret. In Patriam postea redeunti magna mihi intercedebat familiaritas cum Eruditissimo Medico D.* Pitcairnio *& D.* D. Gregorio; *quibus significavi qualem pro Quadraturis Seriem haberet D.* Newtonus, *quam penitùs ipsis ignotam uterq; fatebatur. Post aliquot verò menses narrabat mihi D.* Pitcairnius *D.* Gregorium *Seriem similiter abrumpentem invenisse. Ego nullus dubitans, quin eandem ex duabus prædictis Quadraturis ipsi à me communicatis deduxerit, per Literas D.* Newtonum *rogavi, ut Seriem suam mihi transmittere vellet, ut an eadem esset cum* Gregoriana *perspicerem: Rogatui meo annuit Vir illustrissimus per Literas* 19 Sept. 1688 *datas: Nec mirum si parva esset inter utramq; Seriem discrepantia, cum* Gregorius, *ex duobus illis Exemplis & indicatâ a me Seriei* Newtonianæ *indole,*

suam

suam facile deducere potuisset; quamq; statim in Tractatu D. Pitcairnii De Inventoribus *publicandam curavit. Hanc historiolam Lectoribus impertire æquum videbatur, ut soli* Newtono *Seriem illam tribuendam esse cognoscerent. Satius quidem multo fuisset, si ipse (dum vivus esset)* Gregorius *eandem Orbi Mathematico communicasset, quodq; se facturum promisit per Literas dat.* Londini 10, Oct. 1691. *Me interim in iis hortatus est, ut, si quid haberem ad Memoriam ejus in hoc negotio juvandam, id ego quam citissime ad illum transmitterem; quod sine mora a me rem omnem fideliter ab initio narrante factum erat. Opus enim erat mihi facillimum, utpote qui omnes ejus &* Pitcairnii *Literas hanc materiam spectantes tum apud me habuerim, & adhuc habeo.*

Ego interim (ob plures rationes non jam enumerandas) nihil perquam generale in Quadraturis per hujusmodi Series expectandum fore ratus, ad propriam Methodum promovendam Studia mea convertebam: Nec irritos prorsus fuisse conatus colligere potes ex Tractatu Ann. 1693 *edito, & Specimine in* Actis Philosophicis *Anni* 1697 *de Spatiorum Transcendentium Quadraturis, quæ in* Geometria *omnino tum novæ erant. Ejusdem Anno* 1702, *longe ultra omnium aliarum limites promotæ, Theoremata aliquot generalia in* Act. Phil. *Anni* 1703 *erant publicata. Et magnopere mihi placuisse fateor, cum perciperem, quod prædicta Series* Newtoniana *Casum tantum simplicem Theorematis nostri primi comprehenderet. Integram jam Methodum, cum aliis huic affinibus in sequenti libro explicatam B. Lector inveniet. Et si quidpiam in his ad* Geometriam *promovendam sibi occurrat, tum me finem in his edendis propositum obtinuisse sciat.*

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DE

CALCULO FLUENTIUM:

LIBER PRIMUS.

SECTIO I.

In qua traditur Methodus determinandi Figurarum Curvilinearum Quadraturas.

DESIGNENT z Ordinatam, & y Abſciſſam Curvæ propoſitæ, itaq; in omnibus Curvis Algebraicis relatio inter z & y exprimetur per hanc æquationem, ſcil. $z^m = ay^n + bz^e y^r + b^2 z^s y^t + b^3 z^p y^q$ &c. Ubi m, e, s, p, &c. deſignant exponentes quantitatis z, & n, r, t, q, &c. exponentes quantitatis y; & a, b, b^2, b^3, &c. ſunt terminorum coefficientes determinatæ; literæ autem b exponentes 2, 3, 4, &c. non denotant quantitatis b poteſtates Algebraicas, ſed tantum diverſarum literarum locum ſupplent; cavendum itaq; eſt ne in multiplicatione vel diviſione fiat horum exponentium additio vel ſubſtractio: Hanc autem expreſſionem diverſarum literarum per unam literam cum exponentibus 1, 2, 3, 4, &c. uſus non ſpernendos habere ſæpiſſime invenies, ad Seriei progreſſionem detegendam. Deniq; ſit Area quæſita $Azy + Bz^f y^g + Cz^h y^i + Dz^k y^l$ &c. $= F:z\dot{y}$ in qua f, h, k, &c. denotant exponentes quantitatis

titatis z, & g, i, l, &c. quantitatis y exponentes, ac A, B, C, D, &c. terminorum coefficientes determinatas. Ut ergo obtineatur Quadratura Figuræ cujusvis propositæ, duo sunt præstanda; primò ut Exponentes f, g, h, i, &c. Secundò ut coefficientes A, B, C, &c. determinentur: Adeoq; Methodus hæc in duas partes distribuetur.

Methodi pars prior: Exponentes f, g, h, i, &c. determinare. (1.) Ex æquatione propositam Curvam definiente inveniatur valor quantitatis $\dot{z}$ per Calculum fluxionum, e quo exterminetur z^{m-1} per communem Algebram. (2.) Neglectis coefficientibus A, B, C, &c. ut pote hanc investigationem non ingredientibus inveniatur valor quantitatis $\dot{z}$ per Calculum fluxionum, ex æquatione Quadraturam definiente. (3.) Fiat æquatio inter hos duos valores quantitatis $\dot{z}$, quæ a fractis per multiplicationem liberata ita reducatur ut omnes termini sint nihilo æquales, & hæc vocabitur *Æquatio resultans*. (4.) Fiat comparatio (binos & binos comparando) inter exponentes terminorum hujus æquationis resultantis. (5.) Quia diversis modis comparationes illæ institui possunt, ideo rejiciantur omnes, quæ vel ducunt ad absurdum, vel ad exponentium f; g, h, i, &c. valores determinatos.

EXEMPLUM I.

Definiantur Curvæ per æquationem trium terminorum (quæ vocentur *Trinomiales*) tum omnes comprehendentur sub hac generali. $\dot{z}^m = ay^n + bz^e y^r$. Unde erit $Azy + Bz^f y^g = F:z\dot{y}$. Quadratura omnium, quarum Areæ per duos tantùm terminos exprimi possunt. Ex utraq; æquatione inveniatur valor quantitatis $\dot{z}$, & per Regulas in Articulis, 1, 2 & 3 invenietur resultans

$$y^n + z^e y^r + z^{f-1} y^{g+n-1} + z^{e+f-1} y^{g+r-1} = 0.$$

Comparando termini secundi $z^e y^r$ exponentes, cum exponentibus termini tertii (per Art. 4.) erit $e = f - 1$, $r = g + n - 1$, unde $f = e + 1$, & $g = r - n + 1$. Quare si brevitatis causa ponatur $c = r - n$, erit $g = c + 1$, & $r = c + n$: Adeoq; $z^m = ay^n + bz^e y^{c+n}$ Definiet omnes Curvas trinomiales; & $Azy + Bz^{e+1} y^{c+1} = F:z\dot{y}$. Quando Quadratura duobus tantum constet terminis.

Sed si instituta fuisset comparatio inter exponentes termini secundi $z^e y^r$ & termini quarti $z^{e+f-1} y^{g+r-1}$, tum foret $e = e + f - 1$, & $r = g + r - 1$, unde $f = 1$, $g = 1$ scil. indeterminati æquales determinatis, & proinde rejicienda est hæc comparatio, per Art. 5.

EXEMPLUM II.

Definiantur Curvæ per æquationem quatuor terminorum (quæ vocentur *Quadrinomiales*) tum omnes comprehendentur sub hac generali $z^m = ay^n + bz^e y^r + b^2 z^s y^t$. Et in his, quarum Quadraturæ duobus tantum terminis constant, erit $Azy + Bz^f y^g = F: z\dot{y}$. Per calculum fluxionum invenietur (juxta Art. 1.) ex priore

$$\dot{z} = \frac{nay^{n-1} + rbz^e y^{r-1} + tb^2 z^s y^{t-1}}{maz^{-1} y^n + \overline{mb - eb} \times z^{e-1} y^r + \overline{m - s} \times b^2 z^{s-1} y^t} \times \dot{y}$$

Et (juxta Art. 2.) invenietur ex posteriori per Calculum fluxionum

$$\dot{z} = \frac{\overline{1 - A} \times z - gBz^f y^{g-1}}{Ay + fBz^{f-1} y^g} \times \dot{y}.$$ Unde per Art. 3. erit

$$\frac{nay^{n-1} + rbz^e y^{r-1} + tb^2 z^s y^{t-1}}{maz^{-1} y^n + \overline{m - e} \times bz^{e-1} y^r + \overline{m - s} \times b^2 z^{s-1} y^t}$$

$$= \frac{\overline{1 - a} \times z - gBz^f y^{g-1}}{Ay + fBz^{f-1} y^g}.$$ Liberetur hæc æquatio a fractis, & fiant omnes termini nihilo æquales, & quoniam hæc operatio solos exponentes respicit; ideo, neglectis coefficientibus, resultans erit $y^n + z^e y^r + z^s y^t + z^{f-1} y^{g+n-1} + z^{e+f-1} y^{g+r-1} + z^{s+f-1} \times y^{g+t-1} = 0$.

Jam comparando terminum secundum $z^e y^r$ cum termino quarto $z^{f-1} y^{g+n-1}$, erit $e = f - 1$, $r = g + n - 1$, unde $f = e + 1$, $g = r - n + 1$. Et comparando terminum tertium $z^s y^t$ cum quinto $z^{e+f-1} y^{g+r-1}$ erit $s = e + f - 1 = 2e$, $t = g + r - 1$. Ergo si brevitatis causa ponatur $r - n = c$, seu $r = c + n$, erit $g =$

$g = c + 1$, $t = 2c + n$. Et proinde $z^m = ay^n + bz^e y^{c+n} + b^2 \times z^{2e} y^{2c+n}$: Et $Azy + Bz^{e+1} y^{c+1} = F:z\dot{y}$, quando Area per duos terminos exprimitur.

Et ſi Area tribus conſtet terminis $Azy + Bz^f y^g + Cz^h y^i = F:\dot{z}y$ invenietur per ſimilem Calculum $f = e + 1$, $g = c + 1$, $h = 2e + 1$, $i = 2c + 1$; & ſi quatuor conſtant terminis $Azy + Bz^f y^g + Cz^h y^i + Dz^k y^l = F:z\dot{y}$ erit $k = 2e + 1$, $l = 2c + 1$, ut in præcedenti.

Nec-non ſi Curvæ propoſitæ definiantur per æquationem quinq; terminorum $z^m = ay^n + bz^e y^r + b^2 z^s y^t + b^3 z^p y^q$, invenietur per Calculum jam expoſitum (poſito $r - n = c$) $s = 2e$, $t = 2c + n$, $p = 3e$, $q = 3c + n$.

COROLLARIUM I.

$z^m = ay^n + bz^e y^{c+n} + b^2 z^{2e} y^{2c+n} + b^3 z^{3e} y^{3c+n} + b^4 z^{4e} y^{4c+n}$ &c. Eſt æquatio definiens omnes Curvas Algebraicas, quarum Quadraturæ per hanc Methodum Geometricè determinari poſſunt. Non quod omnes figuræ ſint Geometricè quadrabiles, quarum Curvæ definiuntur per æquationes ſub hac generali incluſas; ſed quod conditio ſit neceſſaria Quadrabilitatis, ut Curvarum æquationes contineantur ſub tali generali, in qua quantitatis z exponentes ſint in hac progreſſione arithmetica; e, $2e$, $3e$, $4e$, $5e$, &c. & quantitatis y exponentes in hac n, $c + n$, $2c + n$, $3c + n$, $4c + n$, $5c + n$, &c.

COROLLARIUM II.

$Azy + Bz^{e+1} y^{c+1} + Cz^{2e+1} y^{2c+1} + Dz^{3e+1} y^{3c+1} + Ez^{4e+1} \times y^{4c+1}$ &c. $= F:z\dot{y}$. Eſt expreſſio generalis Quadraturæ, quando Curvæ propoſitæ æquatio continetur ſub æquatione generali in *Corollario* 1. exhibita.

Methodi pars poſterior Coefficientes A, B, C, &c. determinare. Ex valoribus quantitatis z deductis ex æquationibus in *Coroll.* 1 & 2, exhibitis invenietur *Æquatio reſultans*, ut in Art. 3. oſtenſum eſt. Et æquationis hujus uniuſcujuſq; termini coefficiens ponatur nihilo æqualis; unde novæ orientur æquationes, quæ poſt debitam reductionem dabunt quæſitos valores quantitatum A, B, C, &c. & ſic innoteſcet Series, quæ Quadraturam quæſitam determinabit. *Q. E. I.*

SECTIO

SECTIO II.

In qua traditur Methodus inveniendi Conditiones Quadrabilitatis.

QUoniam plures orientur æquationes, quam quæ (juxta Methodi part. poſt.) ſufficiunt ad determinandas coefficientes A, B, C, &c. ideo ex reliquarum reductione invenientur Quadrabilitatis conditiones. *Q. E. I.*

COROLLARIUM I.

Tot ſemper ſunt Quadrabilitatis conditiones, quot ſuperſunt æquationes, poſtquam (per Methodi partem poſteriorem) determinantur A, B, C, &c. & de his conditionibus notandum, quod una reſpiciat exponentes m, n, e, c; reliquæ verò coefficientes a, b, b^2, b^3, &c. reſpiciant.

EXEMPLUM I.

Invenire conditionem Quadrabilitatis pro curvis *Trinomialibus* $z^m = ay^n + bz^e y^{c+n}$, quando Area figuræ tribus conſtat terminis ſcil. $Azy + Bz^{e+1} y^{c+1} + Cz^{2e+1} y^{2c+1} = F:z\dot{y}$. Ex priori invenies per Calculum fluxionum, $\dot{z} = \dfrac{nay^{n-1} + \overline{c+n} \times bz^e y^{c+n-1}}{mz^{m-1} - ebz^{e-1} y^{c+n}} \dot{y}$

Et exterminando z^{m-1} per communem Algebram, invenietur valor quantitatis $\dot{z} = \dfrac{nay^{n-1} + \overline{c+n} \times bz^e y^{c+n-1}}{maz^{-1} y^n + \overline{m-e} \times bz^{e-1} y^{c+n}} \dot{y}$

Similiter ex æquatione Aream quæſitam conſtituente invenietur.

$$\dot{z} = \frac{\overline{1-A}z + \overline{c+1} \times -Bz^{e+1} y^c + \overline{2c+1} \times -Cz^{2e+1} y^{2c}}{Ay + \overline{e+1}Bz^e y^{c+1} + \overline{2e+1}\, Cz^{2e} y^{2c+1}} \dot{y}$$

Unde

Unde ex utroq; erit $\dfrac{nay^{n-1} + \overline{c+n} \times bz^{e}y^{c+n-1}}{maz^{-1}y^{n} + \overline{m-e} \times bz^{e-1}y^{c+n}}$ æqualis

$$\frac{1 - A \times z + \overline{c+1} \times -Bz^{e+1}y^{c} + \overline{2c+1} \times -Cz^{2e+1}y^{2c}}{Ay + \overline{e+1} \times Bz^{e}y^{c+1} + \overline{2e+1} \times Cz^{2e}y^{2c+1}}.$$

Hæc æquatio post debitam reductionem dabit illam quæ resultans vocatur.

$$\left.\begin{matrix} maA \\ -ma \\ naA \end{matrix}\right\} y^{n} \; \left.\begin{matrix} + \overline{c+n} \times bA \\ na \times \overline{e+1} \times B \\ \overline{m-e} \times \overline{A-1} \times b \\ ma \times \overline{c+1} \times B \end{matrix}\right\} z^{e}y^{c+n} \; \left.\begin{matrix} + \overline{c+n} \times \overline{e+1} \times bB \\ na \times \overline{2e+1} \times C \\ \overline{m-e} \times \overline{c+1} \times bB \\ ma \times \overline{2c+1} \times C \end{matrix}\right\} z^{2e}y^{2c+n} \times$$

$$\left.\begin{matrix} + \overline{c+n} \times \overline{2e+1} \times bC \\ + \overline{m-e} \times \overline{2c+1} \times bC \end{matrix}\right\} z^{3e}y^{3c+n} = 0.$$

Ponantur trium primorum terminorum coefficientes nihilo æquales, erit $maA - ma + naA = 0$, $\overline{c+n} \times bA + na \times \overline{e+1} \times B + \overline{m-e} \times \overline{A-1} \times b + ma \times \overline{c+1} \times B = 0$. $\overline{c+n} \times \overline{e+1} \times bB + na \times \overline{2e+1} \times C + \overline{m-e} \times \overline{c+1} \times bB + ma \times \overline{2c+1} \times C = 0$. Ex reductione harum trium æquationum invenientur valores cogniti quantitatum incognitarum A, B, C. Et quia superest ad huc alia æquatio ponendo ultimi termini coefficientem nihilo æqualem scil. $\overline{c+n} \times \overline{2e+1} + \overline{m-e} \times \overline{2c+1} = 0$, ideo hæc æquatio ostendit quænam debeat esse exponentium relatio, ut Curvæ *Trinomialis* Quadratura Geometrica per tres terminos exhibeatur, id est, hæc æquatio dat Quadrabilitatis conditionem quæsitam. *Q. E. I.*

EXEMPLUM II.

Invenire conditiones Quadrabilitatis pro Curvis *Quadrinomialibus* quarum Quadraturæ per tres terminos exprimuntur.

Ex æquatione $z^{m} = ay^{n} + bz^{e}y^{c+n} + b^{2}z^{2e}y^{2c+n}$, & æquatione Quadraturam constituente scil. $Azy + Bz^{e+1}y^{c+1} + Cz^{2e+1}y^{2c+1} = F : zy$, invenietur æquatio resultans per calculum in præcedenti explicatum scil.

+

$$\left.\begin{array}{l} +maA \\ -ma \\ +naA \end{array}\right\} y^{n} \quad \left.\begin{array}{r} +\overline{c+n}\,bA \\ na\times\overline{e+1}\,B \\ \overline{m-e}\times\overline{A-1}\,b \\ ma\times\overline{c+1}\,B \end{array}\right\} z^{e}y^{c+n} \quad \left.\begin{array}{r} +\overline{c+n}\times\overline{e+1}\,bB \\ na\times\overline{2e+1}\,C \\ ma\times\overline{2c+1}\,C \\ \overline{m-e}\times\overline{c+1}\,bB \\ \overline{2c+n}\times b^{2}A \\ \overline{m-2e}\times\overline{A-1}\times b^{2} \end{array}\right\} z^{2e}y^{2c+n}$$

$$+\left.\begin{array}{r} +\overline{c+n}\times\overline{2e+1}\,bC \\ \overline{m-e}\times\overline{2c+1}\,bC \\ \overline{2c+n}\times\overline{e+1}\,b^{2}B \\ \overline{m-2e}\times\overline{c+1}\,b^{2}B \end{array}\right\} z^{3e}y^{3c+n} \quad \left.\begin{array}{r} +\overline{2c+n}\times\overline{2e+1}\times b^{2}C \\ \overline{m-2e}\times\overline{2c+1}\times b^{2}C \end{array}\right\} z^{4e}y^{4c+n}$$

$= 0.$

Ex tribus primis æquationis hujus terminis invenientur valores incognitarum A, B, C; & quia duæ adhuc ſuperſunt, ideo duæ ſunt Quadrabilitatis conditiones; quarum altera reſpicit relationem Exponentium, & habetur ex ultimi termini coefficiente $\overline{2c+n}\times\overline{2e+1} + \overline{m-2e}\times\overline{2c+1} = 0$. Altera vero coefficientes a, b, b^2 reſpicit, & invenietur ex æquatione inter valorem Quantitatis C ex termino tertio deductum & valorem ejuſdem deductum ex termino quarto. Sed ex termino tertio invenietur

$$C = \frac{\overline{m-2e}\times\overline{1-A}+\overline{2c+n}\times -A}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times\frac{b^{2}}{a}$$

$$+\frac{\overline{m-e}\times\overline{c+1}+\overline{c+n}\times\overline{e+1}}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times -\frac{bB}{a}.$$ Et ex quarto

$$C = \frac{\overline{m-2e}\times\overline{c+1}+\overline{2c+n}\times\overline{e+1}}{\overline{m-e}\times\overline{2c+1}+\overline{c+n}\times\overline{2e+1}}\times -\frac{b^{2}B}{b}.$$

Unde ex utraq; erit

$$\frac{\overline{m-2e}\times\overline{1-A}+\overline{2c+n}\times -A}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times\frac{b^{2}}{a}$$

$$+ \frac{\overline{m-e} \times \overline{c+1} + \overline{c+n} \times \overline{e+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\frac{bB}{a} \text{ æqualis}$$

$$\frac{\overline{m-2e} \times \overline{c+1} + \overline{2c+n} \times \overline{e+1}}{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}} \times -\frac{b^2 B}{b}.$$

Si in hac æquatione pro A, & B ſubſtituantur earum valores, qui facilè inveniuntur ex coefficientibus primi & ſecundi termini æquationis reſultantis, habebitur altera Quadrabilitatis conditio, id eſt, ut Figura propoſita ſit Geometricè Quadrabilis, oportet ut coefficientes a, b, b^2 habeant relationem quæ per æquationem ultimam aſſignatur.

COROLLARIUM II.

Sit N numerus terminorum æquationis Curvas propoſitas definientis, erit $\frac{Nc - 2c + 2e - Ne + m + n}{-cm - en}$ illa Quadrabilitatis conditio, quæ reſpicit exponentes; nimirum debet numerus per hanc quantitatem Algebraicam expreſſus eſſe integer & poſitivus: Unde in Curvis *Trinomialibus* debet $\frac{c - e + m + n}{-cm - en}$, eſſe numerus integer & poſitivus; nam in his $N = 3$; ſimiliter in Curvis Quadrinomialibus debet $\frac{2c - 2e + m + n}{-cm - en}$ eſſe numerus integer & poſitivus, nam in his $N = 4$. Et ſic in cæteris.

COROLLARIUM III.

Quando figuræ propoſitæ ſunt Geometricè Quadrabiles, tum $\frac{Nc - 2c + 2e - Ne + m + n}{-cm - en} + 1$, eſt numerus terminorum (ab initio ſumptorum) Seriei, qui Quadraturam quæſitam conſtituunt. Quoniam in æquationibus Curvas definientibus, z & y exponentes obtinent indefinitos, ideo neceſſe eſt ut Quadraturæ exprimantur per Series infinitas, ut in *Coroll.* 2. *Sect.* 1. Sed ſi per Methodum in hac ſectione explicatam conſtet Figuram propoſitam eſſe Geometricè quadrabilem, tum definitus Seriei terminorum numerus Quadraturam illius Geometricam conſtituet; eſtq; numerus iſte, qualis in hoc *Corollario* aſſignatur.

ſignatur. Sin Figura propoſita non ſit Quadraturæ Geometricæ capax, tum ipſa Series infinita Figuræ iſtius Quadraturam exhibebit.

COROLLARIUM IV.

N — 2 eſt numerus conditionum Quadrabilitatis, quarum una ſemper reſpicit exponentes; reliquæ verò ſpectant ad coefficientes. Poſſunt quidem omnes hæ conditiones quæ coefficientes reſpiciunt ſub una involvi; ſed neceſſaria illa coefficientium a, b, b^2, b^3, &c. relatio multo faciliùs apprehendetur, quando conditiones illæ in plures reſolvuntur; quales contingunt ex comparatione terminorum æquationis reſultantis.

SCHOLIUM.

Quia diverſis modis comparari poſſunt (juxta Regulam in *Partis* 1, *Art.* 4.) exponentes terminorum æquationis reſultantis, ad determinandos, f, g, h, i, &c. ut jam notavi in *Art.* 5. Methodi *Partis* 1. *Sect.* 1. Et quia plures ex iſtis modis diverſis exhibeant expreſſiones Quadraturæ diverſas; quibus proinde diverſæ competunt Quadrabilitatis conditiones; ideo hanc materiam in hoc Scholio explicare viſum fuit, eſt enim in hoc de Quadraturis negotio magni momenti, quamvis illam ne adnotârunt quidem hactenus Geometræ, utpote Methodis utentes non eo uſque pertingentibus.

Reſumatur itaq; *Exemplum* 1. in *Sect.* 1. $z^m = ay^n + bz^e y^r$, & $Azy + Bz^f y^g = F:z\dot{y}$. Quando Area duobus tantum terminis conſtat ex his invenietur reſultans.

$$\left.\begin{matrix} maA \\ -ma \\ naA \end{matrix}\right\} y^n \quad \left.\begin{matrix} +rbA \\ +mbA \\ -ebA \\ -mb \\ +eb \end{matrix}\right\} z^e y^r \quad \left.\begin{matrix} +nfaB \\ +mgaB \end{matrix}\right\} z^{f-1} y^{g+n-1}$$

$$\left.\begin{matrix} +rfbB \\ +mgbB \\ -egbB \end{matrix}\right\} z^{e+f-1} y^{g+r-1} = 0.$$

Jam non tantum comparari poſſunt exponentes termini ſecundi & tertii, ut in *Sect.* 1, & ex qua invenimus $f = e + 1$, $g = r - n + 1$, ſeu (poſito $c = r - n$) $g = c + 1$: Adeoq; $Azy + Bz^{e+1} y^{c+1} = F:z\dot{y}$: Sed etiam comparari poſſunt exponentes terminorum pri-

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mi & ultimi; unde $n = g + r - 1$, $e + f - 1 = 0$; adeoq; $f = 1 - e$, $g = n - r + 1 = 1 - c$, qui dant $Azy + Bz^{1-e}y^{1-c} = F:z\dot{y}$. Duæ itaq; sunt expressiones Quadraturæ; & sicut per Methodum in hac Sectione explicatam inventa est $\overline{c+n} \times \overline{e+1} + \overline{m-e} \times \overline{c+1} = 0$, conditio Quadrabilitatis primæ expressioni competens; sic per eandem methodum invenies, quod $n \times \overline{1-e} + m \times \overline{1-c} = 0$, sit Quadrabilitatis conditio quæ convenit expressioni secundæ.

Et quidem notatu dignum est diversas illas Quadraturarum expressiones diversis æquationis Curvas propositas definientis formis convenire, quod per Exemplum sequens illustrabitur.

Sit $z^5 = ay^{11} + bz^2y^5$; hæc divisa per z^2 ad hanc secundam formam $z^3 = by^5 + az^{-2}y^{11}$ reducetur. Jam pro priori forma $Azy + Bz^{e+1}y^{c+1} = F:z\dot{y}$; & $Azy + Bz^{1-e}y^{1-c} = F:z\dot{y}$; ut ex conditionibus utriq; formæ assignatis patebit.

SECTIO III.

In qua traduntur Theoremata ex Methodo precedenti deducta.

I. *Theorema* generale, quod exhibeat Quadraturas omnium Figurarum, quarum Curvæ definiuntur per æquationem trium terminorum. Esto $z^m = ay^n + bz^e y^{c+n}$ *Æquatio generalis*; quæ definiet omnes Curvas *Trinomiales*. Sed per *Coroll.* 2. *Sect.* 1.

$F:z\dot{y} = Azy + Bz^{e+1}y^{c+1} + Cz^{2e+1}y^{2c+1} + Dz^{3e+1}y^{3c+1} + Ez^{4e+1}y^{4c+1}$ &c. Et per regulas in *Part.* 2. *Sect.* 1. traditas invenientur A, B, C, &c. scil.

$$A = \frac{m}{m+n}.$$

$$B = \frac{\overline{m-e} \times \overline{1-A} + \overline{c+n} \times \overline{-A}}{m \times \overline{c+1} + n \times \overline{e+1}} \times \frac{b}{a}$$

$$C = \frac{\overline{m-e} \times \overline{c+1} + \overline{c+n} \times \overline{e+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\frac{bB}{a}.$$

$D =$

$$D = \frac{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\frac{bC}{a}$$

$$E = \frac{\overline{m-e} \times \overline{3c+1} + \overline{c+n} \times \overline{3e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\frac{bD}{a}$$

$$F = \frac{\overline{m-e} \times \overline{4c+1} + \overline{c+n} \times \overline{4e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\frac{bE}{a}.$$

Ex his ſex terminis manifeſta eſt Seriei progreſſio in infinitum; adeoq; innoteſcit Quadratura generalis omnium figurarum, quarum Curvæ ſunt *Trinomiales*. Et hæ figuræ Quadraturam admittunt Geometricam, quoties $\frac{c-e+m+n}{-cm-en}$ eſt numerus integer ac poſitivus: Et hæc Quadratura Geometrica habetur ſumendo tot, ab initio, terminos, quot ſunt unitates in numero per $\frac{c-e+m+n}{-cm-en}+1$, deſignato.

DEFINITIONES.

Ex æquationibus Curvas definientibus (quæ includuntur ſub æquatione generali exhibitâ in *Coroll.* 1. *Sect.* 1.) illa vocetur *Completa*, in quibus Quantitatis z exponentes ſunt in hac progreſſione Arithmetica e, $2e$, $3e$, $4e$, &c. & exponentes quantitatis y in hac n, $c+n$, $2c+n$, $3c+n$, &c. Sed ſi deſint unus pluresve termini intermedii, tum vocabitur *Æquatio deficiens*.

II. *Theorema* exhibens Quadraturas omnium Figurarum, quarum Curvæ definiuntur per æquationem completam quatuor terminorum, ſcil. $z^m = ay^n + bz^e y^{c+n} + b^2 z^{2e} y^{2c+n}$.

Jam per *Coroll.* 2. *Sect.* 1. Quadraturæ quæſitæ exprimentur ut in *Theoremate* I. Et ſi determinentur A, B, C, &c. per regulas in *Sect.* 1. traditas, invenies.

$$A = \frac{m}{m+n} \qquad B = \frac{\overline{m-e}\times\overline{1-A}-cA-nA}{m\times\overline{c+1}+n\times\overline{e+1}}\times\frac{b}{a}$$

$$C = \frac{\overline{\overline{m-2e}\times\overline{1-A}+\overline{2c+n}\times -A}\times\frac{b^2}{a}+\overline{\overline{c+n}\times\overline{e+1}+\overline{m-e}\times\overline{c+1}}\times -\frac{bB}{a}}{m\times\overline{2c+1}+n\times\overline{2e+1}}$$

$$D = \frac{\overline{\overline{m-2e}\times\overline{c+1}+\overline{2c+n}\times\overline{e+1}}\times -\frac{b^2B}{a}+\overline{\overline{m-e}\times\overline{2c+1}+\overline{c+n}\times\overline{2e+1}}\times -\frac{bC}{a}}{m\times\overline{3c+1}+n\times\overline{3e+1}}$$

$$E = \frac{\overline{\overline{m-2e}\times\overline{2c+1}+\overline{2c+n}\times\overline{2e+1}}\times -\frac{b^2C}{a}+\overline{\overline{m-e}\times\overline{3c+1}+\overline{c+n}\times\overline{3e+1}}\times -\frac{bD}{a}}{m\times\overline{4c+1}+n\times\overline{4e+1}}$$

$$F = \frac{\overline{\overline{m-2e}\times\overline{3c+1}+\overline{2c+n}\times\overline{3e+1}}\times -\frac{b^2D}{a}+\overline{\overline{m-e}\times\overline{4c+1}+\overline{c+n}\times\overline{4e+1}}\times -\frac{bE}{a}}{m\times\overline{5c+1}+n\times\overline{5e+1}}$$

Ex his ſex terminis patet Seriei progreſſio in infinitum, adeoque Quadraturam exhibuimus generalem omnium figurarum, quarum Curvæ definiuntur per æquationem completam quatuor terminorum. Et in his duæ ſunt Quadrabilitatis conditiones; quarum altera poſtulat ut $\frac{2c - 2e + m + n}{-cm - en}$ ſit numerus integer & poſitivus; altera vero quæ ſpectat ad coefficientes a, b, b^2 invenienda eſt ut oſtenſum eſt in *Sect.* 2. Et quando hæ figuræ ſunt Geometricè Quadrabiles, tum tot Seriei termini ab initio ſumpti (quot ſunt unitates in numero $\frac{2c - 2e + m + n}{-cm - en} + 1$) Quadraturam quæſitam conſtituent.

III. *Theorema* exhibens Quadraturas omnium Figurarum, quarum Curvæ definiuntur per æquationem completam quinque terminorum ſcil. $z^m = ay^n + bz^e y^{c+n} + b^2 z^{2e} y^{2c+n} + b^3 z^{3e} y^{3c+n}$.

Jam per *Coroll.* 2. *Sect.* 1. Quadraturæ quæſitæ exprimentur per $F: z\dot{y} = Azy + Bz^{e+1}y^{c+1} + Cz^{2e+1}y^{2c+1} + Dz^{3e+1}y^{3c+1} + {}$ $+ Ez^{4e+1}y^{4c+1}$ &c.

Et per Regulas in *Sect.* 1. invenientur A, B, C, &c. ſcilicet

$A =$

$$A = \frac{m}{m+n}$$

$$B = \frac{\overline{m-e} \times \overline{1-A-cA-nA}}{m \times \overline{c+1} + n \times \overline{e+1}} \times \frac{b}{a}$$

$$C = \left\{ \begin{array}{l} \dfrac{\overline{m-2e} \times \overline{1-A} + \overline{2c+n} \times -A}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times \dfrac{b^2}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{c+1} + \overline{c+n} \times \overline{e+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\dfrac{bB}{a} \end{array} \right.$$

$$D = \left\{ \begin{array}{l} \dfrac{\overline{m-3e} \times \overline{1-A} + \overline{3c+n} \times -A}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times \dfrac{b^3}{a} \\[2ex] \dfrac{\overline{m-2e} \times \overline{c+1} + \overline{2c+n} \times \overline{e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\dfrac{b^2 B}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\dfrac{bC}{a} \end{array} \right.$$

$$E = \left\{ \begin{array}{l} \dfrac{\overline{m-3e} \times \overline{c+1} + \overline{3c+n} \times \overline{e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{b^3 B}{a} \\[2ex] \dfrac{\overline{m-2e} \times \overline{2c+1} + \overline{2c+n} \times \overline{2e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{b^2 C}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{3c+1} + \overline{c+n} \times \overline{3e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{bD}{a} \end{array} \right.$$

$$F = \left\{ \begin{array}{l} \dfrac{\overline{m-3e} \times \overline{2c+1} + \overline{3c+n} \times \overline{2e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{b^3 C}{a} \\[2ex] \dfrac{\overline{m-2e} \times \overline{3c+1} + \overline{2c+n} \times \overline{3e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{b^2 D}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{4c+1} + \overline{c+n} \times \overline{4e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{bE}{a}. \end{array} \right.$$

Ex his ſex terminis manifeſta eſt progreſſio reliquorum: Adeoq; exhibita eſt Quadratura generalis omnium Figurarum, quarum Curvæ definiuntur per *Æquationem completam* quinque terminorum. In his tres ſunt Quadrabilitatis conditiones, quarum una eſt, ut $\frac{2c - 2e + m + n}{-cm - en}$ ſit numerus integer & poſitivus; duæ reliquæ (quæ ſpectant ad coefficientes a, b, b^2, b^5) inveniuntur ut in *Sect.* 2. Deniq; $\frac{2c - 2e + m + n}{-cm - en} + 1$ eſt numerus terminorum Seriei, qui Quadraturam conſtituunt, quando Figura Quadraturam admittit Geometricam.

IV. *Theorema* exhibens Quadraturas omnium Figurarum, quarum Curvæ definiuntur per *Æquationem completam* ſex terminorum ſcil. $z^m = ay^n + bz^e y^{c+n} + b^2 z^{2e} y^{2c+n} + b^3 z^{3e} y^{3c+n} + b^4 z^{4e} y^{4c+n}$. Per Calculum in *Sect.* 1. traditum invenies A, B, C, D, &c. ſcil.

$$A = \frac{m}{m+n}$$

$$B = \frac{\overline{m-e} \times \overline{1-A} + \overline{c+n} \times -A}{m \times \overline{c+1} + n \times \overline{e+1}} \times \frac{b}{a}$$

$$C = \left\{ \begin{array}{l} \dfrac{\overline{m-2e} \times \overline{1-A} + \overline{2c+n} \times -A}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\dfrac{b^2}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{c+1} + \overline{c+n} \times \overline{e+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\dfrac{bB}{a} \end{array} \right.$$

$$D = \left\{ \begin{array}{l} \dfrac{\overline{m-3e} \times \overline{1-A} + \overline{3c+n} \times -A}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times \dfrac{b^3}{a} \\[2ex] \dfrac{\overline{m-2e} \times \overline{c+1} + \overline{2c+n} \times \overline{e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\dfrac{b^2 B}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\dfrac{bC}{a} \end{array} \right.$$

$E =$

$$E = \left\{ \begin{array}{l} \dfrac{\overline{m-4e}\times\overline{1-A}+\overline{4c+n}\times\overline{-A}}{m\times\overline{4c+1}+n\times\overline{4e+1}}\times\dfrac{b^4}{a} \\[2ex] \dfrac{\overline{m-3e}\times\overline{c+1}+\overline{3c+n}\times\overline{e+1}}{m\times\overline{4c+1}+n\times\overline{4e+1}}\times-\dfrac{b^3B}{a} \\[2ex] \dfrac{\overline{m-2e}\times\overline{2c+1}+\overline{2c+n}\times\overline{2e+1}}{m\times\overline{4c+1}+n\times\overline{4e+1}}\times-\dfrac{b^2C}{a} \\[2ex] \dfrac{\overline{m-e}\times\overline{3c+1}+\overline{c+n}\times\overline{3e+1}}{m\times\overline{4c+1}+n\times\overline{4e+1}}\times-\dfrac{bD}{a} \end{array} \right.$$

$$F = \left\{ \begin{array}{l} \dfrac{\overline{m-4e}\times\overline{c+1}+\overline{4c+n}\times\overline{e+1}}{m\times\overline{5c+1}+n\times\overline{5e+1}}\times-\dfrac{b^4B}{a} \\[2ex] \dfrac{\overline{m-3e}\times\overline{2c+1}+\overline{3c+n}\times\overline{2e+1}}{m\times\overline{5c+1}+n\times\overline{5e+1}}\times-\dfrac{b^3C}{a} \\[2ex] \dfrac{\overline{m-2e}\times\overline{3c+1}+\overline{2c+n}\times\overline{3e+1}}{m\times\overline{5c+1}+n\times\overline{5e+1}}\times-\dfrac{b^2D}{a} \\[2ex] \dfrac{\overline{m-e}\times\overline{4c+1}+\overline{c+n}\times\overline{4e+1}}{m\times\overline{5c+1}+n\times\overline{5e+1}}\times-\dfrac{bE}{a}. \end{array} \right.$$

Ex his ſex terminis patet reliquorum, in infinitum, progreſſio; adeoq; exhibita eſt Quadratura generalis omnium Figurarum, quarum Curvæ definiuntur per *Æquationem completam* ſex terminorum. In his quatuor ſunt Quadrabilitatis conditiones, quarum una eſt, ut $\dfrac{4c-4e+m+n}{-cm-en}$ ſit numerus integer & poſitivus; & reliquæ tres inveniuntur per regulas in *Sect.* 2. traditas. Deniq; $\dfrac{4c-4e+m+n}{-cm-en}$ $+ 1$ eſt numerus terminorum Seriei, qui Quadraturas Geometricas conſtituunt.

SCHO-

SCHOLIUM.

Si diligenter obſervetur compoſitio valorum, quos obtinent quantitates A, B, C, &c. in quatuor *Theorematis* jam adductis, facilè percipietur compoſitio valorum earundem, quando *Æquatio* Curvas definiens conſtat Septem, Octo, Novem, vel quotcunq; terminis. Hiſce itaq; ulteriùs explicandis immorari non erit opus, tum jam tot Methodi hujus *Exempla* adduxerim, ut ſine ullo Calculi Labore exhiberi poſſunt hujuſmodi Quadraturæ generales, pro Curvis quæ definiuntur per *Æquationes* utcunq; compoſitas.

Corollarium generale.

Invenire Quadraturas Figurarum, quarum Curvæ definiuntur per *Æquationem* quamlibet deficientem.

Deſtruantur omnes illi termini, quos terminorum deficientium coefficientes b, b^2, b^3, &c. ingrediuntur in valoribus quantitatum A, B, C, &c. computatis ex *Æquatione* completa propoſitam *Æquationem deficientem* includente; reliqui horum valorum termini erunt valores quantitatum A, B, C, &c. pro *Æquatione deficiente* propoſita.

EXEMPLUM I.

Invenire Quadraturas omnium Figurarum, quarum Curvæ definiuntur per hanc *Æquationem deficientem*

$$z^m = ay^n + bz^e y^{c+n} + b^3 z^{3e} y^{3c+n}.$$

Quia hæc ſub completa *Theorematis* III. includitur, & quia hìc deeſt terminus $b^2 z^{2e} y^{2c+n}$, ideo ex *Theoremate* III. deſtruo omnes terminos, quos b^2 ingreditur: Unde pro deficiente propoſita erit.

$A =$

$$A = \frac{m}{m+n}$$

$$B = \frac{\overline{m-e} \times \overline{1-A} + \overline{c+n} \times -A}{m \times \overline{c+1} + n \times \overline{e+1}} \times \frac{b}{a}$$

$$C = \frac{\overline{c+n} \times \overline{e+1} + \overline{m-e} \times \overline{c+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\frac{bB}{a}$$

$$D = \begin{cases} \dfrac{\overline{m-3e} \times \overline{1-A} + \overline{3c+n} \times -A}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times \dfrac{b^3}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\dfrac{bC}{a} \end{cases}$$

$$E = \begin{cases} \dfrac{\overline{m-3e} \times \overline{c+1} + \overline{3c+n} \times \overline{e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{b^3B}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{3c+1} + \overline{c+n} \times \overline{3e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{bD}{a} \end{cases}$$

$$F = \begin{cases} \dfrac{\overline{m-3e} \times \overline{2c+1} + \overline{3c+n} \times \overline{2e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{b^3C}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{4c+1} + \overline{c+n} \times \overline{4e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{bE}{a} \end{cases}$$

EXEMPLUM II.

Invenire Quadraturas omnium Figurarum, quarum Curvæ definiuntur per hanc *Æquationem deficientem*

$$z^m = a y^n + b z^e y^{c+n} + b^4 z^{4e} y^{4c+n}.$$

Quia hæc includitur sub completa, ex qua *Theorema* IV. deducitur; & quia hic desunt $b^2 z^{2e} y^{2c+n}$, & $b^3 z^{3e} y^{3c+n}$, ideo ex illo *Theoremate* destruo omnes terminos, quos b^2 & b^3 ingrediuntur: Unde pro deficiente proposita

$$A = \frac{m}{m+n}$$

$$B = \frac{\overline{m-e} \times \overline{1-A} + \overline{c+n} \times -A}{m \times \overline{c+1} + n \times \overline{e+1}} \times \frac{b}{a}$$

$$C = \frac{\overline{m-e} \times \overline{c+1} + \overline{c+n} \times \overline{e+1}}{m \times \overline{2c+1} + n \times \overline{2e+1}} \times -\frac{bB}{a}$$

$$D = \frac{\overline{m-e} \times \overline{2c+1} + \overline{c+n} \times \overline{2e+1}}{m \times \overline{3c+1} + n \times \overline{3e+1}} \times -\frac{bC}{a}$$

$$E = \begin{cases} \dfrac{\overline{m-4e} \times \overline{1-A} + \overline{4c+n} \times -A}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times \dfrac{b^4}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{3c+1} + \overline{c+n} \times \overline{3e+1}}{m \times \overline{4c+1} + n \times \overline{4e+1}} \times -\dfrac{bD}{a} \end{cases}$$

$$F = \begin{cases} \dfrac{\overline{m-4e} \times \overline{c+1} + \overline{4c+n} \times \overline{e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{b^4B}{a} \\[2ex] \dfrac{\overline{m-e} \times \overline{4c+1} + \overline{c+n} \times \overline{4e+1}}{m \times \overline{5c+1} + n \times \overline{5e+1}} \times -\dfrac{bE}{a}. \end{cases}$$

Conditiones Quadrabilitatis, & numerus terminorum Seriei, qui constituunt Geometricam Quadraturam Figurarum, quarum Curvæ definiuntur per *Æquationes deficientes* coincidunt cum iis, quæ conveniunt *Æquationibus* illis completis, quæ deficientes includunt.

SECTIO

SECTIO IV.

In qua traduntur Regulæ applicandi Theoremata præcedentia ad Figuras particulares.

1. REducatur *Æquatio* Curvam particularem definiens ita, ut unus terminus ex sola z constare possit, sitq; ille terminus altera pars *Æquationis*, omnibus reliquis terminis ad alteram partem translatis.

2. *Æquationis* sic reductæ termini comparentur cum terminis *Æquationis* istius generalis, quæ propositam particularem comprehendit; ex hac comparatione invenientur exponentium m, n, c, e, & coefficientium a, b, b^2, b^3, &c. valores.

3. Substituantur hi valores in illo *Theoremate* generali, quod deducitur ex illa *Æquatione* quacum proposita fuit comparata; & sic habebitur Quadratura Figuræ particularis propositæ.

EXEMPLUM I.

Sit $z^3 + y^3 = bzy$: Seu $z^3 = -y^3 + bzy$: Ex comparatione hujus cum Trinomio generali erit $m = 3$, $n = 3$, $e = 1$, $c = -2$, $a = -1$, $b = b$; & quia hi exponentes efficiunt ut $\frac{c - e + m + n}{-cm - en}$ sit numerus integer & positivus scil. 1; ideo figura proposita est Geometricè Quadrabilis, & quia $\frac{c - e + m + n}{-cm - en} + 1 = 2$, ideo si in duobus primis terminis *Theor.* 1. substituantur valores quantitatum m, n, e, c, a, b modo assignati, habebis figuræ propositæ Quadraturam scil. $F : z\dot{y} = \frac{1}{2} zy - \frac{1}{6} bz^2 y^{-1}$.

EXEMPLUM II.

Sit $z^3 = ay + bz^{-18} y^3$, ex comparatione hujus cum generali in *Art.* 1. *Sect.* 3. erit $m = 3$, $n = 1$, $e = -18$, $c = 2$; & quia ex his $\frac{c - e + m + n}{-cm - en} = 2$, ideo Figura proposita Quadraturam admit-

admittit Geometricam; adeoq, si in *Theor.* 1. substituantur modo assignati valores quantitatum m, n, e, c, habebitur Figuræ propositæ Quadratura scil. $F:z\dot{y} = \frac{3}{4}zy - \frac{3b}{8a}z^{-17}y^{3} - \frac{9bb}{40aa}z^{-35}y^{5}$.

EXEMPLUM III.

Sit $z^{5} = p^{4}y + q^{12}z^{-9}y^{2}$; ex comparatione hujus cum generali erit $m = 5$, $n = 1$, $e = -9$, $c = 1$, $a = p^{4}$, $b = q^{12}$. Sed $\frac{c - e + m + n}{-cm - en} = 4$; & proinde si in quinq; primis terminis *Theor.* 1. substituantur modo assignati valores quantitatum m, n, e, c, a, b, erit $F:z\dot{y} = \frac{5}{6}zy + \frac{1}{3}q^{12}p^{-4}z^{-8}y^{2} + 2q^{24}p^{-8}z^{-17}y^{3} + \frac{8q^{36}y^{4}}{3p^{12}z^{26}} + \frac{16q^{48}y^{5}}{15p^{16}z^{35}}$.

EXEMPLUM IV.

Sit $z^{\frac{5}{3}} = ay^{\frac{1}{4}} - pz^{7}y^{\frac{3}{2}}$; ex comparatione hujus cum Trinomio generali erit $m = \frac{5}{3}$, $n = \frac{1}{4}$, $e = 7$, $c = \frac{5}{4}$, $b = -p$; & quia $\frac{c - e + m + n}{-cm - en} = 1$, ideo Figura proposita est Geometricè Quadrabilis; & Quadratura habetur substituendo valores quantitatum m, n, e, c, b in duobus primis terminis *Thear.* 1. scil. $F:z\dot{y} = \frac{20}{23}zy + \frac{184pz^{8}y^{\frac{9}{4}}}{\overline{23}|^{2}a}$.

EXEMPLUM V.

Sit $z^{2} = ay^{-1} + 2az^{3}y + az^{6}y^{3}$; ex comparatione hujus cum generali Quadrinomio (ex quo *Theor.* 11. deducitur invenies $m = 2$, $n = -1$, $c = 2$, $e = 3$; unde $\frac{2c - 2e + m + n}{-cm - en} = 1$, quæ est conditio Quadrabilitatis exponentes respiciens; & $b = 2a$, $b^{2} = a$, quibus competunt altera Quadrabilitatis conditio, ut per Methodum in *Sect.* 2. traditam patebit; & proinde Figura proposita est Geome-

Geometricæ Quadraturæ capax: Et quia $\frac{2e - 2e + m + n}{-cm - en} + 1 = 2$, ideo Quadratura illa obtinetur substituendo prædictos valores quantitatum m, n, e, c; b, b^2 in *Theorematis* II. duobus primis terminis; $F : z\dot{y} = 2zy - z^3y$. Q. E. I.

Notandum 1. Quoniam *Æquatio* proposita ad plures formas reduci potest, & quia (ut in *Scholio Sect.* 2. observatum fuit) hæ diversæ formæ diversas postulant Quadraturæ expressiones, & diversas Quadrabilitatis conditiones, ex Methodo præcedenti facilè deducendas, ut in *Scholio* prædicto ostensum fuit; ideo in applicatione generalis cujusvis Quadraturæ ad Figuram particularem datam necesse erit *Æquationem particularem* datam reducere ad illam formam, cui competit illa Quadrabilitatis conditio (exponentes respiciens) quæ assignatur *generali Æquationi* particularem datam includenti: Et si nulla sit hujusmodi forma *Æquationis* propositæ; tum non datur (per hanc saltem Methodum) Figuræ propositæ Quadratura Geometrica.

EXEMPLUM I.

Sit $z^3 = ry^5 + sz^{-2}y^{11}$ *Æquatio* data, quæ comparata cum generali dat $m = 3$, $n = 5$, $e = -2$, $c = 6$, unde $\frac{c - e + m + n}{-cm - en}$ non est numerus integer & positivus, adeoq; sub hac forma non obtinebitur Quadratura figuræ propositæ ex *Theor.* 1: Sed si *Æquatio* data multiplicetur per z^2, tum reducetur ad hanc formam $z^5 = sy^{11} + rz^2y^5$, unde comparando erit $m = 5$, $n = 11$, $e = 2$, $c = -6$; atq; hi valores faciunt $\frac{c - e + m + n}{-cm - en}$ numerum integrum; & positivum, adeoq; sub hac secunda formâ Figuræ propositæ Quadratura obtinebitur ex *Theor.* 1.

EXEMPLUM II.

Sit $z^2 = ry^2 + sz^2y^2$ *Æquatio* data; hæc comparata cum generali dat $m = 2$, $n = 2$, $e = 2$, $c = 0$, unde $\frac{c - n + m + n}{-cm - en}$ non est numerus integer & positivus, adeoq; sub hac formâ deduci non potest Quadratura ex *Theoremate* I. Sed si *Æquatio* data dividatur

tur per z^2, tum erit $z^0 = s y^2 + r z^{-2} y^2$, unde comparando hanc cum generali erit $m = 0$, $n = 2$, $e = -2$, $c = 0$; & hi valores faciunt ut $\frac{c - e + m + n}{-cm - en}$ sit numerus integer ac positivus, & proinde sub hac forma habetur Quadratura ex *Theor.* 1.

Vix opus est, ut moneam diversas illas formas obtineri dividendo *Æquationem datam* per Quantitatis z potestatem in unoquoq; termino: Unde patet *Æquationem* ad tot diversas formas esse reducibilem, quot sunt diversæ potestates quantitatis z in *Æquatione* proposita.

Regulæ inveniendi Æquationem generalem, *quæ quamlibet particularem propositam comprehendet.*

EX *Æquatione generalissimâ* in *Coroll* 1. *Sect.* 1. manifestum est, quod in eodem termino idem sit numerorum e & c multiplus. Jam secundus terminus *Æquationis* quæsitæ semper est $a y^n$, qui respondet illo termino *Æquationis* datæ, in quo sola y (cum determinatis) occurrit, & proinde n est numerus cognitus. Sit p exponens quantitatis z, & q exponens quantitatis y in tertio *Æquationis* datæ termino; p^2, q^2 exponentes earundem in termino quarto; p^3, q^3 in termino quinto; & sic porro; ubi exponentes literarum p, q adhibentur ad denotandos diversos numeros datos. Sit etiam l multiplus numerorum e, c in termino tertio *Æquationis generalis* quæsitæ; l^2 multiplus eorundem in termino quarto; l^3 multiplus in quinto; & sic porro. Unde $l e = p$, & $l c + n = q$, $l^2 e = p^2$, & $l^2 c + n = q^2$, $l^3 e = p^3$, $l^3 c + n = q^3$, & sic porro. Ex debita harum reductione invenies $l = \frac{p}{e}$, $l^2 = \frac{p^2}{e}$, $l^3 = \frac{p^3}{e}$, &c. Sed l, l^2, l^3, &c. Sunt numeri integri & positivi, ergo talis debet esse valor numeri e, ut $\frac{p}{c}$, $\frac{p^2}{e}$, $\frac{p^3}{e}$ &c. sint numeri integri; ex cognito autem e, cognoscuntur c & l, adeoq; & *Æquatio generalis* quæsita. *Q. E. I.*

EXEM-

EXEMPLUM I.

Sit $z^3 = ay^2 + rz^4y^{18} + sz^7y^{30}$. In hoc caſu $m = 3$, $n = 2$, $p = 4$, $p^2 = 7$; $q = 18$, $q^2 = 30$. Jam ut $\frac{p}{e} = \frac{4}{e}$, & $\frac{p^2}{e} = \frac{7}{e}$ ſint numeri integri & poſitivi oportet ut ſit $e = 1$, adeoq; $l = 4$, $l^2 = 7$; unde $c = \frac{q - n}{l} = \frac{18 - 2}{4} = 4$. Si ergo in *Æquatione* data pro z^3 ſubſtituatur z^m & ay^n pro ay^2, & le (ſeu $4e$) pro p (ſeu 4) & $lc + n$ (ſeu $4c + n$) pro q (ſeu 18) item $l^2 e$ (ſeu $7e$) pro p^2 (ſeu 7) & etiam $l^2 c + n$ (ſeu $7c + n$) pro q^2 (ſeu 30;) *Æquatio* erit $z^m = ay^n + rz^{4e}y^{4c+n} + sz^{7e}y^{7c+n}$, quæ eſt generalis quæſita.

EXEMPLUM II.

Sit $z^4 = ay^5 + rz^{\frac{2}{3}}y^{\frac{1}{2}} + szy^{-\frac{7}{4}}$. Ubi $m = 4$, $n = 5$, $le = \frac{2}{3}$, $l^2 e = 1$: Unde $l = \frac{2}{3e}$, $l^2 = \frac{1}{e}$, ut l & l^2 ſint numeri integri & poſitivi, eſto $e = \frac{1}{3}$, unde $l = 2$, $l^2 = 3$: Sed etiam per Regulam precedentem $lc + n = \frac{1}{2}$, ſeu $2c + 5 = \frac{1}{2}$, unde $c = -\frac{9}{4}$; ſi ergo in *Æquatione* data pro 4 & 5 ſubſtituantur m & n, $lc + n$, (ſeu $2c + n$) pro $\frac{1}{2}$, & $l^2 c + n$, (ſeu $3c + n$) pro $-\frac{7}{4}$ erit

$$z^m = ay^n + rz^{2e}y^{2c+n} + sz^{3e}y^{3e+n}. \quad \text{Q. E. I.}$$

N. B. Cum innumeris modis ſumi poſſit (e) tamen quo major eſt numerus, eò ſimplicior erit *Æquatio generalis*; & proinde ſumatur ſemper maximus valor numeri e, ita ut l, l^2, l^3, &c. ſint integri & poſitivi.

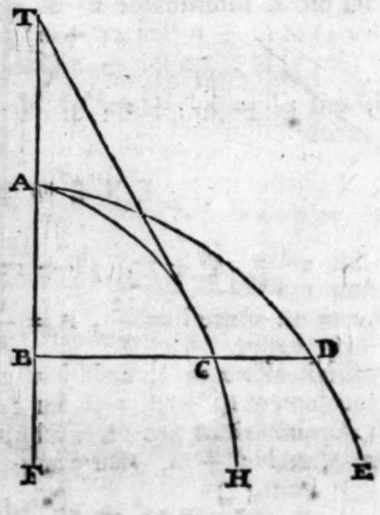
SECTIO

SECTIO V.

De Quadraturis Figurarum Geometricè irrationalium.

HUjusmodi aliquot Quadraturas edidi in *Actis Philosophicis* R. S. Anni 1697. Jam ipsam Methodum (vel potius Methodi cujusdam Elementa) variis *Exemplis* illustratam sum traditurus.

Si in *Æquatione* Curvam ADE definiente occurrat alterius (non rectificabilis) Curvæ ACH arcus $AC = v$, cujus abscissa $AB = y$ eadem sit cum abscissa Curvæ ADE, cujus Ordinata $BD = z$; tum illa Curva vocetur Geometricè irrationalis, vel simpliciter, irrationalis: Et speciatim si ACH sit Curva rationalis seu *Algebraica*, tum Curva ADE est Curva irrationalis primi generis; & ad has solas spectat Methodus jam tradenda; adeoq; $v = AC$ semper est arcus Curvæ rationalis, quæ definitur per *Æquationem*, cujus termini lineis tantum rectis constant.

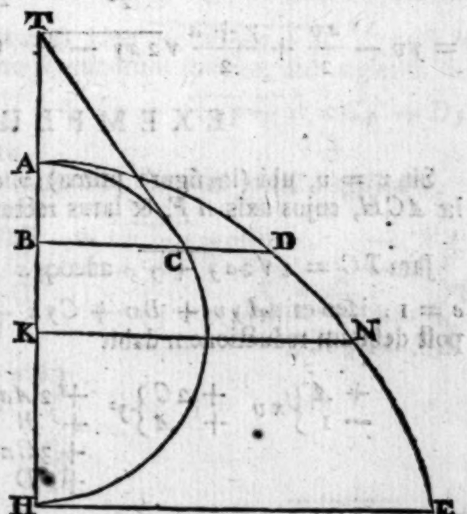


1. Ad Curvæ ACH punctum quodvis C esto tangens CT axi AB occurrens in puncto T. 2. Inveniatur Tangentis CT valor Analyticus per y & datas expressus; & per x denotetur pars istius valoris quæ sub vinculo radicali Quadratico continetur, postquam quantum fieri potest a quantitate y liberatur, ut in Tractatu de Quadraturis Anni 1693, edito ostensum est. 3. Posito quod e sit exponens dimensionum ad quas indeterminatæ z, y, v assurgunt in *Æquatione* Curvam ADE definiente, fiat involutio hujus quantitatis $\overline{v + x + y + 1}|^{e+1}$, & neglectis coefficientibus numericis ex involutione oriundis, afficiantur termini coefficientibus incognitis A, B, C, &c. & summa omnium ponatur quantitati $F : z\dot{y}$ *Æqualis*. 4. Sumetur fluxio hujus *Æquationis*

tionis & eliminentur v, $\dot{x}$; & post alias usitatas reductiones habebitur *Æquatio*, quam voco *resultantem*; cujus termini more usitato comparati dabunt A, B, C, &c. adeoq; $F:z\dot{y}$ seu aream quæsitam. Quoniam *Æquatio* per *Reg.* 3. constituta sæpissimè plurimos comprehendit terminos inutiles, ideo in sequentibus tales ommittuntur, quos Calculum expertus ad Quadraturam non spectare deprehendi.

EXEMPLUM I.

Sit $z = v$, ACH Semicirculus, cujus Diameter $AH = 2a$, invenire Quadraturam Areæ ABD.



In hoc & sequentibus denotentur quantitates ut in præcedenti Figurâ. Jam $TC = \frac{a}{a-y}\sqrt{2ay-y^2}$; unde $x = \sqrt{2ay-y^2}$; & in hoc *Exemplo* $e = 1$. Ergo etiam $\overline{v+x+y+r}|^e$ est quantitas, cujus termini coefficientibus A, B, C, &c. affecti constituent Quadraturam quæsitam $F:z\dot{y}$. Sed, neglectis terminis inutilibus, erit $Ayv + Bv + Cyx + Dx = F:z\dot{y}$. Hujus fluxio per regulas præcedentes reducta dabit

$$\left.\begin{matrix}+A\\-1\end{matrix}\right\}x\dot{v}\quad\left.\begin{matrix}+Aa\\+3Ca\\-D\end{matrix}\right\}y\quad\left.\begin{matrix}+Ba\\+Da\end{matrix}\right\}\quad-2Cy^2\Big\}=0.$$

Ex terminorum comparatione invenientur $A = 1$, $B = -a$, $C = 0$, $D = A$; unde $ABD = yv - av + a\sqrt{2ay-y^2} = F:z\dot{y}$. Q. E. I.

EXEMPLUM II.

Sit ADE Cyclois vulgaris, cujus *Æquatio* eſt $z = v + \sqrt{2ay - y^2}$, ubi v denotat arcum AC Circuli genetoris, cujus Diameter $AH = 2a$; adeoq; $x = \sqrt{2ay - y^2}$.

Jam quia $e = 1$, invenietur ut ſupra $Ayv + Bv + Cyx + Dx = F:z\dot{y}$, & per *Reg.* 4. invenies $A = 1$, $B = -\frac{a}{2}$, $C = \frac{1}{2}$, $D = \frac{a}{2}$, & proindè $ABD = yv - \frac{av}{2} + \frac{yx + ax}{2} = F:z\dot{y}$; id eſt, $ABD = yv - \frac{av}{2} + \frac{y + a}{2}\sqrt{2ay - y^2}$. Q. E. I.

EXEMPLUM III.

Sit $z = v$, ubi (in figura prima) v denotat arcum AC parabolæ ACH, cujus axis AF, & latus rectum $8a$.

Jam $TC = 2\sqrt{2ay + y^2}$, adeoq; $x = \sqrt{2ay + y^2}$. Et quia $e = 1$; ideo erit $Ayv + Bv + Cyx + Dx = F:z\dot{y}$, cujus fluxio poſt debitam reductionem dabit

$$\left.\begin{matrix} +A \\ -1 \end{matrix}\right\} xv \quad \left.\begin{matrix} +2C \\ +A \end{matrix}\right\} y^2 \quad \left.\begin{matrix} +2Aa \\ +B \\ +3Ca \\ +D \end{matrix}\right\} y \quad \left.\begin{matrix} +2Ba \\ +Da \end{matrix}\right\} = 0.$$

Unde $A = 1$, $B = \frac{a}{2}$, $C = -\frac{1}{2}$, $D = -a$, adeoq; Quadratura quæſita $ABD = yv + \frac{1}{2}av - \overline{a - \frac{1}{2}y} \times \sqrt{2ay + y^2} = F:z\dot{y}$.

EXEMPLUM IV.

Sit (in *Fig.* 1.) ADE Curva funicularia in ſitu ad naturalem contrario; tum per conſtructionem *Bernoullianam* erit $z = v - \sqrt{2ay + y^2}$; ubi v eſt arcus AC parabolæ ACH, cujus Axis AF & latus rectum $8a$.

Erit

Erit (ut in præcedenti) $x = \sqrt{2ay + y^2}$; & quia in hoc casu $e = 1$, ideo $Ayv + Bv + Cyx + Dx = F:z\dot{y}$; & determinando A, B, C, D, per *Reg.* 4. $A = 1$, $B = a$, $C = -1$, $D = -2a$. Unde $ABD = yv + av - \overline{y - 2a} \times \sqrt{2ay + y^2}$.

EXEMPLUM V.

Sit (in *Fig.* 2.) $z = ry^n v$, ubi est v arcus Semicirculi ACH, cujus Radius $AH = 2a$.

Quia in hoc casu $x = \sqrt{2ay - y^2}$; & e est numerus indefinitus (ob indefinitum exponentem n) ideo $\overline{v + x + y + 1}|^{e+1}$ post involutionem constabit numero terminorum infinito, sed neglectis superfluis, erit $F:z\dot{y} = ay^{n+1} v + Bv + \sqrt{2ay - y^2} \times Cy^n + Dy^{n-1} + Ey^{n-2} + Fy^{n-3}$ &c.

Cujus fluxio post debitam reductionem juxta *Regulam* 4. erit

$$\left.\begin{array}{r} \overline{n+1} \times A \\ -r \end{array}\right\} xy^n v \left.\begin{array}{r} + Aa \\ - C \\ + nC \end{array}\right\} y^n + \left.\begin{array}{r} \overline{2n+1} \times aC \\ + D \\ \overline{n-1} \times -D \end{array}\right\} y^n \left.\begin{array}{r} - E \\ - aD \\ \overline{n-1} \times 2aE \\ \overline{n-2} \times -E \end{array}\right\} y^{n-1}$$

$$\left.\left.\begin{array}{r} - F \\ + aE \\ \overline{n-2} \times aE \\ \overline{n-3} \times -F \end{array}\right\} y^{n-2} \text{ \&c. } \begin{array}{r} + aB \\ \\ \\ \end{array}\right\} = 0.$$

Unde $A = \dfrac{r}{n+1}$, $C = \dfrac{ra}{\overline{n+1}|^2}$, $D = \dfrac{\overline{2n+1} \times ra^2}{n \times \overline{n+1}|^2}$,

$E = \dfrac{\overline{2n-1} \times aD}{n-1}$, $F = \dfrac{\overline{2n-3} \times aE}{n-2}$ adeoque etiam

$G = \dfrac{\overline{2n-5} \times aF}{n-3}$, $H = \dfrac{\overline{2n-7} \times aG}{n-4}$ &c.

Ex his patet (1.) Quod si exponens n sit numerus integer & positivus, vel etiam si $2n$ sit numerus positivus & impar, tum Area exprimetur per finitum numerum Terminorum seriei; quia in his casibus Series abrumpitur. (2.) Quod $-B$ sit Æqualis coefficienti termini ultimo abrumpentis.

EXEM-

EXEMPLUM VI.

Sit $z = ry^n v^2$, cæteris positis ut in *Exemplo* 5, erit $F:z\dot{y} = Ay^{n+1} v + Bv^2 + v\sqrt{2ay - y^2} \times Cy^n + Dy^{n-1} + Ey^{n-2} + Fy^{n-3} + Gy^{n-4} + Hy^{n-5} + Ky^{n-6}$ &c. $+ Ly^{n+1} + My^n + Oy^{n-1} + Py^{n-2} + Qy^{n-3}$ &c.

Unde $A = \frac{r}{n+1}$, $C = \frac{2ra}{\overline{n+1}|^2}$, $D = \frac{\overline{2n+1} \times 2ra}{n \times \overline{n+1}|^2}$, $E = \frac{\overline{2n-1} \times aD}{n-1}$, $F = \frac{\overline{2n-3} \times aE}{n-2}$, $G = \frac{\overline{2n-5} \times aF}{n-3}$, $H = \frac{\overline{2n-7} \times aG}{n-4}$, $K = \frac{\overline{2n-9} \times aH}{n-5}$ &c.
Adeoque constat progressio terminorum, qui multiplicantur in $v\sqrt{2ay - y^2}$; Similiter ex comparationibus reliquis invenietur $L = -\frac{2raa}{\overline{n+1}|^3}$, $M = -\frac{\overline{2n+1} \times 2ra^3}{n^2 \times \overline{n+1}|^3}$, $O = -\frac{\overline{2n-1} \times aaD}{\overline{n-1}|^2}$, $P = -\frac{\overline{2n-3} \times a^2E}{\overline{n-2}|^2}$, $Q = -\frac{\overline{2n-5} \times aaF}{\overline{n-3}|^2}$; & sic patet progressio terminorum absolutorum; & pro coefficiente B nondum determinata notandum, quod $-2B$ sit Æqualis coefficienti termini ultimò abrumpentis scil. ultimi ex iis qui in $v\sqrt{2ay - yy}$ multiplicantur.

EXEMPLUM VII.

Sit $z = ry^n v$, ubi in *Fig.* 1. v denotat arcum AC parabolæ ACH, cujus Axis AF, & latus rectum $8a$. Erit $F:z\dot{y} = Ay^{n+1} v + Bv + \sqrt{2ay + y^2} \times Cy^{n+1} + Dy^n + Ey^{n-1} + Fy^{n-2} + Gy^{n-3}$, &c.

Unde

Unde $A = \frac{r}{n+1}$, $C = -\frac{r}{n \times \overline{n+1}|^2}$, $D = -\frac{ra}{n+2 \times \overline{n+1}|^2}$;

$E = \frac{\overline{2n+1} \times ra^2}{n \times \overline{n+2} \times \overline{n+1}|^2}$, $F = -\frac{\overline{2n-1} \times aE}{n-1}$, $G = \frac{\overline{2n-3} \times aF}{n-2}$,

adeoq; $H = \frac{\overline{2n-5} \times aG}{n-3}$, &c.

Notandum 1. Quod $-B$ ſit æqualis coefficienti ultimò abrumpentis termini. 2. Quod coefficiens termini ultimi ſemper ſit duplicandus, quando ſit applicatio ad figuram particularem.

EXEMPLUM VIII.

Sit $z = ry^n v^2$, cæteris poſitis ut in *Exemplo* 7, erit $F:zy = Ay^{n+1} v^2 + Bv^2 + v\sqrt{2ay+y^2} \times \overline{Cy^{n+1} + Dy^n + Ey^{n-1} + Fy^{n-2} + Gy^{n-3} + Hy^{n-4} + Ky^{n-5}}$ &c. $+ Ly^{n+3} + My^{n+2} + Ny^{n+1} + Oy^n + Py^{n-1} + Qy^{n-2} + Ry^{n-3}$ &c.

Unde $A = \frac{r}{n+1}$, $C = \frac{-2r}{n+2 \times \overline{n+1}}$, $D = \frac{-2ra}{n+2 \times \overline{n+1}|^2}$,

$E = \frac{\overline{2n+1} \times aD}{n}$, $F = -\frac{\overline{2n-1} \times aE}{n-1}$, $G = \frac{\overline{2n-3} \times aF}{n-2}$,

$H = -\frac{\overline{2n-5} \times aG}{n-3}$, $K = \frac{\overline{2n-7} \times aH}{n-4}$ &c.

Sic patet progreſſio terminorum, qui multiplicantur in $v\sqrt{2ay+yy}$.

Similiter $L = \frac{2r}{\overline{n+1} \times \overline{n+2} \times \overline{n+3}}$, $M = \frac{-2a+D}{n+2}$,

$N = \frac{2aA - E}{n+1}$, $O = \frac{-2aE + F}{n}$, $P = \frac{2aF - G}{n-1}$,

$Q = \frac{-2aF + G}{n-2}$, $R = \frac{2aG - H}{n-3}$, $S = \frac{-2aH + K}{n-4}$, &c.

Notandum. Ex terminis qui multiplicantur in $v\sqrt{2ay + y^2}$, duplicanda est coefficiens abrumpentis, ubicunq; occurrit. 2. — $2B$ est æqualis coefficienti quæ afficit terminum abrumpentem, qui multiplicatur in $v\sqrt{2ay + y^2}$: Sed (in hoc valore quantitatis — $2B$) hæc coefficiens non est duplicanda.

SCHOLIUM.

Ex his Quadraturis irrationalibus facile invenietur an Figuræ portio quævis sit Geometricè Quadrabilis, quænam illa portio sit, quæq; ejus Quadratura. Nam cum termini irrationales (*i. e.* termini quos v ingreditur) impediunt, quo minus Areæ expressio generalis sit Geometrica; ideo ut portio aliqua sit Quadraturæ Geometricæ capax, efficiendum est ut evanescant omnes termini, in quibus v occurrit; quod fit ponendo omnes illos terminos nihilo Æquales: Ex hac *Æquatione* inveniatur quantitatis y valor determinatus; & si hic valor fuerit affirmativus, tum Figuræ propositæ competit specialis Quadratura Geometrica portionis illius, quæ adjacet speciali huic abscissæ y valorem habentis modò inventum. Et si in Areæ expressione generali pro y substituatur valor ejus determinatus, tum termini irrationales evanescent, & reliqui rationales dabunt Geometricam prædictæ portionis Quadraturam.

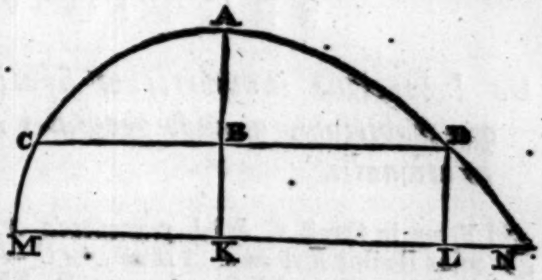
Sic in *Exemplo* 1, si ponatur $yv - av = 0$, erit $y = a$. Unde patet quod si Ordinata KN transeat per Circuli centrum K, tum Area $AKN = aa$, *i. e.* Figuræ irrationalis AEH portio AKN est æqualis Radii quadrato.

Similiter, si in *Exemplo* 2, ponatur $yv - \frac{av}{2}$, erit $y = \frac{a}{2}$. Unde patet, quod si Ordinata Cycloidis BD bisecet radium AK, tum portio ejus $ABD = \frac{3aa}{8}\sqrt{3}$. Et sic in aliis portionem habentibus Quadrabilem.

Ex hoc fundamento deducuntur omnia illa, quæ de Cycloidis Spatiis innumeris Geometricè quadrabilibus tradiderunt fratres Clarissimi *Ja.* & *Jo. Bernouilli* in *Actis Erud.* Lipsiæ. Ipse autem (ni fallor) primus detexi hoc fundamentum in *Actis Phil. R. S. Sept.* 1697, in quibus duo exhibui *Theoremata* innumerarum figurarum irrationalium Quadraturas determinantia, quæ in Geometriâ prorsus nova erant. Utq; ulterius pateat Methodi præcedentis usus ostendam quo pacto illius ope inveniatur Quadratura indefinita & Geometrica Figurarum irrationalium, ex data una tali.

COROL-

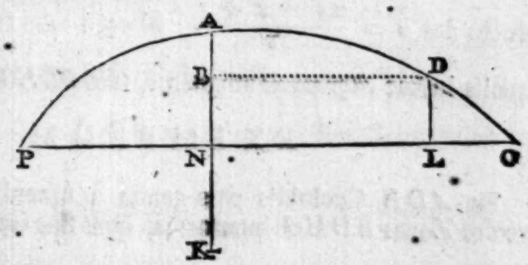
COROLLARIUM I.



Sit *ADN* Curva in *Exemplo* primo proposita, in qua $AC = v$ est arcus Quadrantis *ACM*, cujus centrum *K*, & radius $AK = a$. Adeoque BD, seu $z = v$, ex puncto quovis *D* fit *DL* normalis ad ordinatam *KN* per centrum *K* transeuntem; Jam posito quod $AB = y$, erit $ADLK = a\sqrt{2ay - y^2}$ indefinite. Ut patebit si ad Quadraturam irrationalem in *Exemp.* 1. exhibitam addatur expressio analytica rectanguli $BL = v \times \overline{a - y}$.

Nec non spatium trilineum $DLN = a^2 - a\sqrt{2ay - y^2}$, indefinite; nam $DLN = AKN - ADLK$.

COROLLARIUM II.



Sit *K* centrum, & $Ka = a$ radius arcus *AP*, cujus sinus rectus *PN* bisecat Radium in puncto *N*; sitq; Cycloidis pars, ab arcu *AP* genita *ADO*; sitq; *BD* quælibet Ordinata cujus Abscissa $AB = y$. A puncto *D* sit *DL* ad Ordinatam *NO* perpendicularis, erit $ADLN = \frac{2a - y}{2}\sqrt{2ay - yy}$; Nec non $DLO = \frac{3a^2}{8}\sqrt{3} + \frac{y - 2a}{2}\sqrt{2ay - y^2}$. Demonstratio est similis præcedenti.

Adeoq; duas exhibet Methodus nostra Spatiorum Cycloidalium Quadraturas Geometricas & indefinitas, qualium ne una quidem hactenus innotuit, etiam post tot clarissimorum Geometrarum inquisitiones in notabilis hujus Figuræ Quadraturas. In sectione sequenti celebriora aliquot proferentur *Exempla* ad illustrandam Methodum in *Scholio* præcedenti exhibitam.

EXEM-

SECTIO VI.

De Inventione innumerorum Spatiorum Geometricè quadrabilium, quando generalis Areæ expressio est irrationalis.

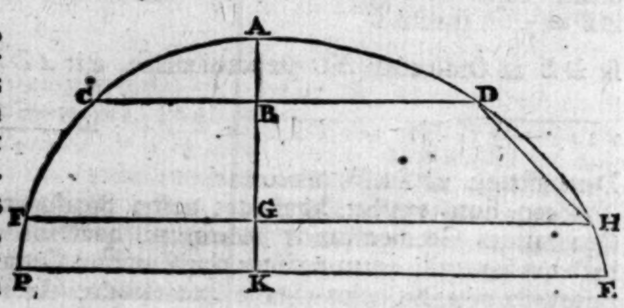
SIT, ut in *Coroll.* 1. *Schol.* præcedentis ACP Quadrans Circuli cujus Radius $KA = a$, $AB = y$, $AC = v$, $AF = nv$; sitq; ADE Curva talis, ut ductis utcunq; CD, FH perpendicularibus ad AK, sint Ordinatæ $BD = AC = v$, $CH = ACF = nv$ (ubi n denotat quemlibet numerum positivum unitate majorem) invenire Zonas innumeras $BDHG$ Geometricè Quadrabilibus.

Sit $AG = x$, $BC = p$, $GF = q$. Tum ex Figuræ hujus Quadratura generali in *Exemplo* 1: *Sect.* 5. erit
$AGH = nxv - nav + aq$. $ABD = yv - av + ap$; Unde $BDHG = nxv - nav + aq - yv + av - ap = AGH - ABD$. Ergo per *Schol. Sect.* 5. $nxv - nav - yv + av = 0$, quæ reducta dat $x = \dfrac{na - a + y}{n}$. Si ergo talis fuerit inter x & y relatio qualis in hac *Æquatione* assignatur, tum $BDHG = a \times \overline{q - p}$. Q. E. I.

EXEMPLUM II.

Sit ADE Cycloidis pars genita a Circuli Quadrante ACP: Invenire Zonas $BDHG$ innumeras, quæ sint Geometricè Quadrabiles.

Ducantur CD, FH ad PE parallelæ secantes Radium AK in B & G. Sit $AG = x$, $GF = q$. $AB = y$, $BC = p$; $AK = a$, $AC = v$, $ACF = nv$; adeoq; ex natura



Cy-

Cycloidis erit $BD = v + p = v + \sqrt{2ay - y^2}$, $GH = nv + q$ $= nv + \sqrt{2ax - x^2}$: Unde ex Cycloidis Quadratura generali in *Exemp.* 2. *Sect.* 5, erit

$$AGH = \frac{2nxv - nav + aq + xq}{2},\ ABD = \frac{2yv - av + ap + yp}{2};$$

$$\text{Unde } BDHG = \frac{2nxv - nav + aq + xq - 2yv + av - ap - yp}{2}$$
$$= AGH - ABD.$$

Ideo per *Schol.* præc. $2nxv - nav - 2yv + av = 0$; unde $x = \frac{na - a + 2y}{2n}$; & quando talis est Abscissarum x, y relatio, qualis hic assignatur, tum $BDHG = \frac{aq + qx - ap - py}{2}$. Q. E. I.

COROLLARIUM.

Ut hæc spatia sint Quadrabilia, oportet ut $y \sqsupset \frac{a}{2}$. Nam ex *Hyp.* $x \sqsubset y$. Ergo $\frac{na - a + 2y}{2n} \sqsubset y$, unde $\frac{a}{2} \sqsubset y$. Q. E. D.

EXEMPLUM III.

Invenire segmenta Cycloidis DH innumera, quæ sint Geometricè quadrabilia.

Quia Trapezium rectilineum $BDHG = \frac{1}{2} BG \times \overline{GH + BD}$ $= \frac{x - y}{2} \times \overline{nv + q + v + p}$; ideo si hæc substrahatur ex valore Zonæ Curvi lineæ $BDHG$ in præcedenti invento erit

$$DH = \frac{nxv - nav - px + aq + nyv - xv - yv + av + qy - ay}{2}$$

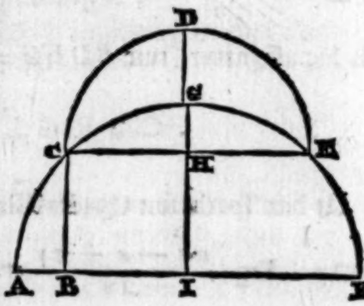
Unde per *Schol.* præcedentem $nxv - nav + nyv - xv - yv + av = 0$, quæ dat $x = \frac{na - a + y - ny}{n - 1} = a - y$. Unde

constat

constat quod si in Axe AK capiatur $KG = AB$; & a punctis B, G ducantur Cycloidis ordinatæ GH, BD, erit (ducta recta à D ad H) Segmentum Cycloidis $DH = \frac{2aq - xq - xp - ap}{2}$.

EXEMPLUM IV.

Sit AGF Semicirculus cujus Diameter AF, cui ex centro I normalis sit IG, & eidem parallela sit Chorda CE secans IG in H: Tum centro H & radio HC describatur Semicirculus CDE: Quæritur punctum C tale, ut Lunula $CDEGC$ sit Geometricè Quadrabilis.



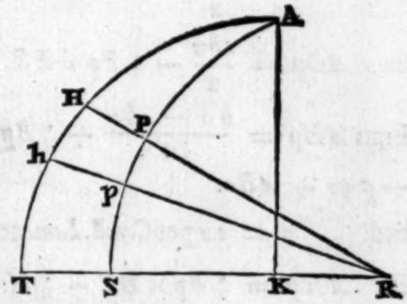
Producatur IG donec Circulo minori occurrat in D: Et a C ducatur CB ad AF normalis. Sit $AI = a$, $IB = HC = b$, $BC = c$, arcus $AC = v$, Quadrans $ACG = nv$; adeoq; Quadrans $CD = \frac{nbv}{a}$.

Jam ex Quadratura Circuli irrationali patet quod $ACGEF = nav$; $ACEF = av + bc$; unde $CGE = nav - av - bc = ACGEF - ACEF$: Sed $CDE = \frac{nbbv}{a} = HC \times CD$; unde Lunula $CDEGC = \frac{nbbv}{a} - nav + av + bc = CDE - CGE$. Ideoq; ut Lunula sit quadrabilis oportet ut $\frac{nbbv}{a} - nav + av = 0$ (per *Schol. Sect.* 5.) unde $nb^2 = na^2 - a^2$: Sed $b^2 = aa - cc$; ergo $na^2 - nc^2 = na^2 - a^2$: Unde $c = \frac{a}{\sqrt{n}}$. Ergo ut destruantur termini irrationales, debet numerus n esse talis, ut cum a sit radius, seu sinus Quadrantis $nv = ACG$, sit etiam $\frac{a}{\sqrt{n}}$ sinus rectus arcus $v = AC$: Sed in unico tantum casu hoc fieri potest, scil. quando $n = 2$, adeoq; arcus CGE est Circuli majoris Quadrans, quo casu punctum quæsitum C bisecat Quadrantem ACG: Et

Et proinde $HC = BC$, i. e. $b = c$. Quo casu Lunula $CDEGC = bc = b^2$. Et sic per Methodum in *Scholio* præmisso traditam ducimur ad Hypocratis Lunulam, ejusq; Quadraturam Geometricam.

LEMMA.

Sit $AHTSA$ Semi-lunula, R centrum, RS radius arcûs interioris APS: K centrum, & KA radius arcûs exterioris AHT. Sitq; Hb pars infinitè parva arcus AHT, & a punctis H, b ad centrum R ducantur rectæ HR, bR secantes arcum AS in punctis P, p, erit $Hb.\ Pp :: \sqrt{2}, 1$. Calculum (utpotè prolixum) non addo.



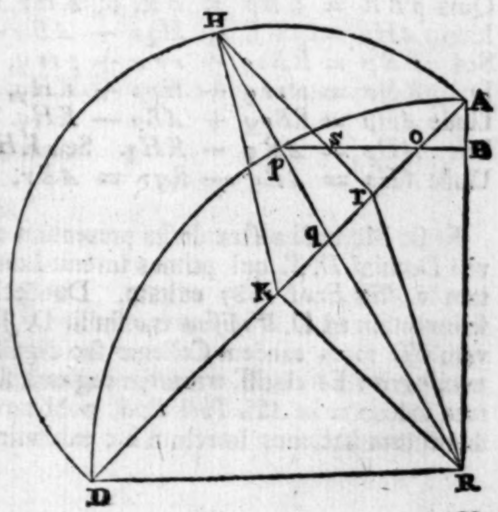
COROLLARIUM.

$AH = AP \times \sqrt{2}$. Nam $Hb.\ Pp :: \sqrt{2}, 1$: Unde etiam $Hb = Pp\sqrt{2}$, & regrediendo ad fluentes $F: Hb = F: Pp\sqrt{2}$; i. e. $AH = AP \times \sqrt{2}$. Q. E. D.

EXEMPLUM V.

Invenire innumeras Lunulæ portiones, quæ sint Geometricè Quadrabiles, nec non earum Quadraturas assignare.

A puncto quovis H sumpto in arcu AHD, ad centrum R arcûs interioris ducatur recta HR secans arcum hunc in p, ejusq; Chordam AD in q. A punctis H, p ducantur Hr, pB ad AD & AR perpendiculares, & se invicem secantes in S; hæc verò secet AD in o. Sitq; $AH = v$, $Ap = u$, $KA = b$, adeoq; $RA = b\sqrt{2}$, seu (posito $e = \sqrt{2}$) $= eb$. Jam quia



$AHp = AHr + rHq - pqA.$
Sed $PqA = pBA + pqo - AoB.$
Ergo $AHP = AHr + rHq - ABp + ABo - pqo.$

Sed $AHr = \frac{bv}{2} - \frac{1}{2} Hr \times rK.$

Et $ABp = \frac{ebu}{2} - \frac{1}{2} Bp \times BR.$

Ergo $AHp = \frac{bv - ebu}{2} + \frac{1}{2} Bp \times BR - \frac{1}{2} Hr \times Kr + Hqr - pqo + ABo.$

Sed $v = ev$ per *Coroll. Lem.* præc. Ergo $\frac{bv - ebu}{2} = 0$; & proinde erit $AHp = \frac{1}{2} Bp \times BR - \frac{1}{2} Hr \times Kr + Hqr - pqo + ABo.$

Et sic invenimus spatium rectilineum spatio Lunulari AHp æquale; unde constat quod quævis recta a Centro R educta abscindat spatium Quadrabile; cujus Quadratura Geometrica hic exhibetur. *Q. E. I.*

SCHOLIUM.

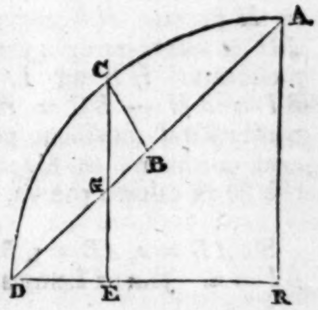
Spatium rectilineum hic exhibitum ad expressionem simpliciorem reduci potest resolvenda triangula in suas partes, hoc modo
Quia $pBR = \frac{1}{2} Bp \times BR$, & $KHr = \frac{1}{2} HR \times KR.$
Ideo $AHp = pBR + Hqr + ABo - KHr - pqo.$
Sed $BRp = RBoq + sro + psrq$, & $Hqr = psrq + Hps.$
Et $KHr = psrq + Hps + KHq$, $pqo = sro + psrq.$
Unde $AHp = RBoq + ABo - KHq$. Sed $RBoq = ARq - ABo.$
Erit $AHp = ARq - KHq.$ Sed $KHq = Rqr.$
Unde $AHp = ARq - Rqr = ARr.$ Q. E. I.

Et sic Methodi nostræ ducta preventum est ad *Theorema* celeberrimi viri Domini *D. T.* qui primus invenit Lunulæ Quadraturam indefinitam in *Actis Erud.* 1687 editam. Duodecim postea annis, similem a se inventam ad D. *Wallisium* transmisit D. *Jo. Perks*: *Wallisius* rem novam esse ratus eandem Collegæ suo dignissimo *D. D. Gregorio* communicavit: Et clariss. triumviri cogitata sua de hac Lunulæ Quadratura indefinita in *Actis Phil. R. S.* publicarunt. Solutionis verò fundamentum hactenus intactum hic exhibuimus.

LEMMA.

LEMMA.

Invenire *Æquationem* per quam puncta Quadrantis ACD referantur ad Chordam AD.



Sit R centrum, RA, RD radii Quadrantis; $AD = a$, $e = \sqrt{2}$, tum a puncto Quadrantis quovis C ducantur CB, CE ad AD & DR perpendiculares; quarum hæc secet AD in G: Et vocetur Abscissa AB, y; ordinata BC, x. Jam

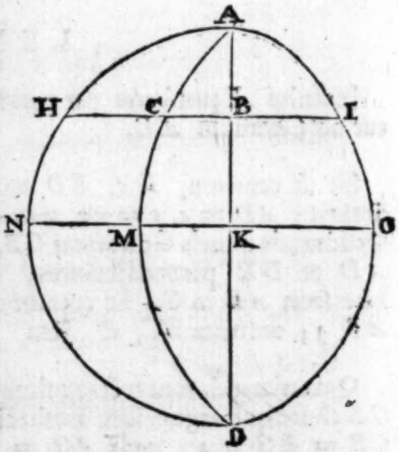
Quia triangulorum rectangulorum CBG, GED reliqui anguli sunt semirecti, ideo $CB = BG = x$; unde $AG = y + x$, adeoq; $DG = a - y - x$; sed $DG^2 = EG^2 + ED^2 = 2ED^2 = 2GE^2$, i. e. $\overline{a-y-x}|^2 = 2GE^2$, seu $GE = \frac{a-y-x}{e}$. Similiter $GC^2 = 2BC^2 = 2x^2$; unde $GC = ex$: Unde $EC = GE + GC = \frac{a-y-x}{e} + ex = \frac{a-y+x}{e}$; sed ex natura Circuli $\overline{2RD - DE} \times DE = EC^2$, & $2RD = ea$, $DE = GE = \frac{a-y-x}{e}$, ideo substituendo hos valores quantitatum $2RD$, DE & EC, invenietur.

$$\frac{a-y-x}{e} \times \frac{a+y+x}{e} = \frac{\overline{a-y+x}|^2}{ee}, \text{ seu } ay - ax = y^2 + x^2.$$

Q. E. I.

PROBLEMA.

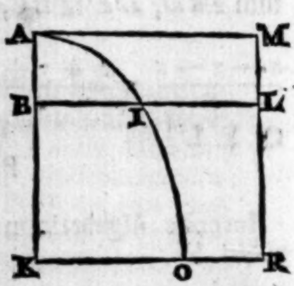
Invenire Algebraicam Curvam, cujus Area sit æqualis Areæ Lunulari.



Sit AD Diameter & K centrum semicirculi AND, quæ cum alterius Circuli quadrante AMB constituit Lunulam $ANDMA$. A semicirculi AND puncto quovis H ducatur HB normalis ad AD & secans quadrantem in C; producatur HB ad I, ut sit $BI = BH - BC = HC$; & puncta illa I transibunt per Curvam quæsitam; & Algebraicam esse sic ex calculo patebit.

Sit $AD = a$, $AB = y$, $BC = x$; $BI = z$. Jam ex Lemmate præcedente erit $x = \frac{\sqrt{a^2 + 4ay - yy} - a}{2}$, $= BC$, & ex natura Circuli $BH = \sqrt{ay - yy}$; Ergo ex constructione $z = \sqrt{ay - yy} - \frac{\sqrt{aa + 4ay - yy} + a}{2}$, quæ definit Curvam $AIOD$, cujus Area $AOD = ANDMA$, & indefinite $ABI = AHC$. Q. E. D.

SCHOLIUM.

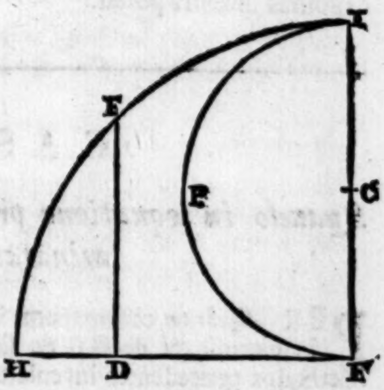


Curva in hoc problemate constructa eadem est cum illa quam exhibuit celeberrimus *Leibnitius* in *Actis Erud. men. Maii* 1684. Sit AIO dimidium Figuræ præcedentis; sitq; AR quadratum a Latere $AK = \frac{1}{2} AD = \frac{1}{2} a = b$; producatur ordinata BI donec Lateri RM occurrat in L; & sit Abscissa $RL = u$, ordinata $LI = w$, unde $z = b - w$, $y = b - u$; itaq; pro z & y substituantur $b - w$, & $b - u$ in æquatione, quam in problemate præcedenti ostendimus definire Curvam AOD; & post debitam hujus reductionem invenietur, $b^4 + w^4 = 6 b^2 w^2 - 4 w^2 u^2$. Eadem scil.

fcil. cum æquatione per quam Curvam fuam definivit *Leibnitius.* Et quia AOK eft æqualis femilunulæ $ANMA = \frac{1}{2} bb$, neq; tum cognita fuit aliarum femilunulæ portionum Quadraturæ, adeoq; nec Figuræ hujus AOK; ideo exinde videtur *Leibnitius* fuam de Quadraturis in generales & fpeciales diftinctionem deduxiffe.

EXEMPLUM VI.

Sit F centrum Quadrantis IEH, ejufq; radius FI fit femicirculi IPF Diameter: Invenire in Quadrante punctum E, a quo ductus finus rectus ED cum cofinu DF & arcu IE & femicirculo IPF comprehendant fpatium quadrabile.



Sit $IE = u$, $IPF = v$, $ED = b$, $DF = c$, $FI = a$. Jam ex Circuli quadratura irrationali, erit Area $IFDE = \frac{au + bc}{2}$. Et $IPF = \frac{av}{4}$, & proinde erit $IPFDEI = \frac{au + bc}{2} - \frac{av}{4}$; unde per *Schol.* *Sect.* 5. $\frac{au}{2} - \frac{av}{4} = 0$, adeoque $u = \frac{1}{2} v$: Sed ex Elementis notum eft quod $IPF = IEH$; ergo $u = \frac{1}{2} IEH$, quare punctum quæfitum E bifecat Quadrantem IEH: Et in hoc cafu erit $IPFDEI = \frac{bc}{2} = \frac{bb}{2} = \frac{a^2}{4}$. Q. E. I.

SECTIO

SECTIO VII.

In qua traduntur Calculi Fluentium Regulæ quædam generales.

QUando per *Algebram* vulgarem *Æquatio Fluxionalis* ita reduci potest, ut quantitates fluxionales $\dot{x}$, $\dot{y}$ &c. a suis tantum indeterminatis x, y &c. cum determinatis afficiantur, tum Fluens obtinebitur per Quadraturas; & proinde Fluens ut inventa supponitur, quando Fluxiones cum suis indeterminatis ita a se invicem separantur, quoniam tum per Quadraturarum Methodos in præcedentibus expositas inveniri potest.

CASUS I.

Quando in æquatione proposita deest altera indeterminatarum x, y.

PER *Algebram* communem inveniatur valor istius fluxionis cujus indeterminata deest; & sic separabuntur fluxiones, adeoq; per Methodos præcedentes invenietur fluens.

EXEMPLUM I.

Sit $a^3 \dot{y} = x^3 \dot{x} - a x^2 \dot{y}$: Quia hic deest y, ideo per *Algebræ* regulas quæratur valor fluxionis $\dot{y}$, invenies utiq; $\dot{y} = \frac{x^3 \dot{x}}{a^3 + a x^2}$: Et sic separantur indeterminatæ cum suis fluxionibus.

EXEMPLUM II.

Sit $\frac{a a \dot{x}^2}{y} - 2 b \dot{y} \dot{x} = a \dot{y}^2$: Quia in hac æquatione non occurrit x, ideo quæratur valor fluxionalis quantitatis $\dot{x}$, & per *Algebræ* regulas

gulas ex æquatione propofita invenietur $\dot{x} = \frac{\dot{y}}{aa}\sqrt{b^2y^2 + a^3y} + \frac{by\dot{y}}{aa}$; unde feparantur indeterminatæ cum fuis fluxionibus, adeoq; per Methodos præcedentes invenietur fluens. *Q. E. I.*

CASUS II.

$$ay^m\dot{y} = by^{m+1}\dot{x} + q\dot{x}.$$

In quo q denotat quantitatem ex determinatis & x utcunq; compofitam. Sit fluens quæfita $By^{m+1} = z^e \times F: z^n q\dot{x}$ (unde $Bz^{-e}y^{m+1} = F: z^n q\dot{x}$) cujus fluxio eft $\overline{m+1}\, By^m\dot{y} = z^{e+n}q\dot{x} + ez^{e-1}\dot{z} \times F: z^n q\dot{x}$; in qua fi fubftituatur præcedens valor quantitatis $F: z^n q\dot{x}$, *Æquatio* erit, $\overline{m+1}\, By^m\dot{y} = eBy^{m+1}z^{-1}\dot{z} + z^{e+n}q\dot{x}$. Fiat comparatio inter terminos hujus & *Æquationis* propofitæ, erit (1ª) $\overline{m+1}\, By^m \times \dot{y} = ay^m\dot{y}$; unde $B = \frac{a}{m+1}$ (2ª) $eBy^{m+1}z^{-1}\dot{z} = by^{m+1}\dot{x}$; unde $eBz^{-1}\dot{z} = b\dot{x}$; & quia hic feparantur indeterminatæ cum fuis fluxionibus ideo per Methodos præcedentes invenietur hujus fluens feu z per x & determinatas. (3ª) $z^{e+n}q\dot{x} = q\dot{x}$, unde $z^{e+n} = 1$. Ergo $e + n = 0$; & quia non plures fuperfunt comparationes, ideo fumatur e vel n ad arbitrium; ut fi fumatur $e = 1$, tum $n = -1$; ex his patet quod $\frac{ay^{m+1}}{m+1} = z \times F: z^{-1}q\dot{x}$ fit fluens affumpta, in qua fi pro z fubftituatur valor ejus per x & datas expreffus (per comparationem fecundam inventus) tum fiet feparatio indeterminatarum, & proinde fluens quantitatis $z^{-1}q\dot{x}$ dabitur per Methodos præcedentes: Et fi exterminantur z fluens affumpta dabit relationem inter y & x. Q. E. I.

CASUS III.

$$ay^m\dot{y} = by^{m+1}p\dot{x} + q\dot{x}.$$

In quo p & q funt quantitates ex x determinatis utcunq; compofitæ. Affumo (ut in præcedenti) $By^{m+1} = z^e \times F: z^n q\dot{x}$, cujus fluxio

 eft

est $\overline{m+1} \times By^m \dot{y} = eBy^{m+1} z^{-1} \dot{z} + z^{e+n} q\dot{x}$. Ex comparatione hujus cum proposita erit, $\overline{m+1}\ By^m \dot{y} = ay^m \dot{y}$, unde $B = \frac{a}{m+1}$; & $eBy^{m+1} z^{-1} \dot{z} = by^{m+1} p\dot{x}$ unde $eBz^{-1} \dot{z} = bp\dot{x}$; quia p (ex hypothesi) componitur ex x & determinatis, ideo in hac secunda comparatione separantur indeterminatæ z, x cum suis fluxionibus $\dot{z}$, $\dot{x}$, adeoq; inveniri potest z per x & datas. Deniq; $z^{e+n} q\dot{x} = q\dot{x}$; unde $z^{e+n} = 1$. Adeoq; $e + n = 0$, sumatur ad arbitrium $e = 1$, adeoq; $n = -1$. Ex his valoribus quantitatum B, e, n; fluens assumpta erit $\frac{ay^{m+1}}{m+1} = z \times F: z^{-1} q\dot{x}$, in qua si pro z substituatur ejus valor ex comparatione secunda inventus, & tum dabitur fluens quantitatis $z^{-1} q\dot{x}$; adeoq; habetur fluens quæsita seu *Æquatio* exprimens relationem inter y & x. Q. E. I.

CASUS IV.

$$a\dot{y} = py\dot{x} + by^n q\dot{x}.$$

In quo p & q sunt quantitates compositæ ex x & determinatis; transmutetur proposita *Æquatio* in simpliciorem hoc modo. Assumatur $y = v^r$ ubi v denotat quantitatem indeterminatem cujus exponens est r; ope hujus exterminetur y & $\dot{y}$ ex æquatione proposita, & erit $arv^{r-1}\dot{v} = pv^r\dot{x} + bv^{rn} q\dot{x}$, quæ divisa per v^{rn} dat $arv^{r-1-rn}\dot{v} = pv^{r-rn}\dot{x} + bq\dot{x}$, ut hæc fiat simplicissima ponatur $r - 1 - rn = 0$; unde $r = \frac{1}{1-n}$; adeoq; erit $\frac{a\dot{v}}{1-n} = pv\dot{x} + bq\dot{x}$: Sed manifestum est hanc *Æquationem* contineri sub æquatione casûs 3, scil. quando $m = 0$. Ergo per Methodum in casu 3, traditam invenietur *Æquationis* istius fluens, seu relatio inter v & x: Sed ex assumpta $y = v|^{\frac{1}{1-n}}$ habetur relatio inter v & y: Ergo ex utraq; habebitur relatio inter y & x. Q. E. I.

CASUS V.

$$a y^{c} \dot{y} = b y^{n} q \dot{x} + p \dot{x}.$$

In quo p & q denotant quantitates datas per x; transmutetur *Æquatio* proposita in simpliciorem ut in præcedenti casu, scil. assumendo $y = v^{r}$, cujus ope si exterminetur y & $\dot{y}$, erit $ar \times \overline{v}|^{cr+r-1} \dot{v} = b \overline{v}|^{rn} q \dot{x} + p \dot{x}$. Destruatur exponens quantitatis v in primo termino, ponendo $cr + r - 1 = 0$, seu $r = \frac{1}{c+1}$: Unde $\frac{a\dot{v}}{c+1} = b \overline{v}|^{\frac{n}{c+1}} q \dot{x} + p \dot{x}$; cujus fluens facile invenietur per Methodum quæ in casu 2 & 3 traditur; ex inventa relatione inter v & x, & assumpta relatione ($y = \overline{v}|^{\frac{1}{c+1}}$) inter v & y, habetur relatio inter y & x. Q. E. I.

SCHOLIUM.

Quando per has Methodos *Æquationis* propositæ fluens non obtinetur, tum per illustrissimi *Newtoni* Methodos exprimetur terminorum Serie infinita.

DE
CALCULI FLUENTIUM USU:

LIBER SECUNDUS.

SECTIO I.

Methodus inveniendi Logarithmos.

PER literam l numero cuilibet præfixam denotetur iſtius numeri Logarithmus: Sic $l:3$, ſignificat Logarithmum numeri 3; & $l:\overline{a+1}$ ſignificat numeri cujuſcunq; $a+1$ Logarithmum; ſit ergo x Logarithmus numeri cujuſvis $a+1$. Jam Logarithmus quæſitus x duabus diverſis Methodis inveniri poteſt, quarum prior vocetur *Directa*; quoniam in illâ Logarithmum ex numero immediatè deducere oſtenditur. Altera verò vocetur *Methodus indirecta*, quoniam in illa Logarithmus quæſitus x non immediate ex numero $a+1$ deducitur. Sed ex alterius numeri Logarithmo, per hanc Methodum indirectam inveniendo.

Methodus Directa.

Ex hypotheſi $x = l:\overline{a+1}$; & habeatur hæc *Æquatio Canon generalis.* (1.) Fiat quælibet *Æquatio* inter terminos ex a & y utcunq; compoſitos, & cum aliis quibuſvis Numeris quovis modo combinatos, ſcil. per additionem, ſubſtractionem, multiplicationem, diviſionem

aut radicum extractionem. (2.) Ope *Æquationis* sic ad arbitrium assumptæ exterminetur a ex Canone generali, & sic habebitur *Æquatio* exprimens relationem inter indeterminatas x, y. (3.) Hujus *Æquationis* inveniatur fluxio, & fluxionis sic inventæ exhibeatur fluens per seriem infinitam; hæc series infinita erit numeri propositi $a + 1$ Logarithmus quæsitus x.

EXEMPLUM I.

Assumatur $a = y$; unde per Canonem generalem erit $x = l : \overline{1 + y}$, cujus fluxio est $\dot{x} = \frac{\dot{y}}{1 + y}$; & hujus fluens per Seriem infinitam expressa est

$$x = y - \tfrac{1}{2} y^2 + \tfrac{1}{3} y^3 - \tfrac{1}{4} y^4 + \tfrac{1}{5} y^5 - \tfrac{1}{6} y^6 + \tfrac{1}{7} y^7 \text{ \&c.}$$

Unde patet Logarithmum sic inventum exhibere Hyperbolæ Quadraturam, & proinde Methodus hæc detegit relationem inter Spatia hyperbolica & Logarithmos, jam dudum Geometris cognitam.

EXEMPLUM II.

Assumatur $y = \frac{a}{a + 2}$. Unde $a + 1 = \frac{1 + y}{1 - y}$; Ergo per Canonem generalem $x = l : \frac{1 + y}{1 - y}$; cujus fluxio est $\dot{x} = \frac{2\dot{y}}{1 - yy}$. Et hujus fluens per Seriem expressa est

$$x = 2 \times y + \tfrac{1}{3} y^3 + \tfrac{1}{5} y^5 + \tfrac{1}{7} y^7 + \tfrac{1}{9} y^9 + \tfrac{1}{11} y^{11} \text{ \&c.}$$

EXEMPLUM III.

Assumatur $y = \frac{ra}{na + 2n}$, ubi r, n denotant numeros ad arbitrium sumendos. Ex hac invenies $a + 1 = \frac{r + ny}{r - ny}$; Unde per Canonem generalem $x = l : \frac{r + ny}{r - ny}$; cujus fluxio dabit $\dot{x} = \frac{2nr\dot{y}}{rr - n^2y^2}$. Et hujus fluens per Seriem infinitam expressa est

$$x = 2 \times \frac{ny}{r} + \frac{n^3y^3}{3r^3} + \frac{n^5y^5}{5r^5} + \frac{n^7y^7}{7r^7} \text{ \&c.}$$

LEMMA

LEMMA I.

Sit z Logarithmus cujusvis Fractionis $\frac{b}{a+1}$, x Denominatoris $a+1$ Logarithmus; tum $l:b - z = x$.

LEMMA II.

Sit e Exponens cujusvis Radicis numeri cujuscunque n, erit $l:\sqrt[e]{n} = \frac{l:n}{e}$. Constat utrumq; *Lemma* ex natura Logarithmorum.

Methodus Indirecta.

Sit $a+1$ Numerus primus cujus quæritur Logarithmus x; sit b numerus productus ex multiplicatione numerorum, quorum maximus sit minor quam $a+1$. Sitq; z Logarithmus Fractionis $\frac{b}{a+1}$. Id est, $z = l:\frac{b}{a+1}$. Et hanc *Æquationem* voco Canonem generalem Secundum.

Tum pro b sumatur numerus ex (a) & numeris cognitis utcunq; compositus, tales autem ut eorum maximus sit minor quam $a+1$. Quando in hunc modum $\frac{b}{a+1}$ per a & numeros datos vel assumptos exprimitur, tum (ut in Methodo directâ) assumatur quælibet *Æquatio* inter y & (a) cum numeris quibusvis; & ope hujus *Æquationis* exterminetur (a) ex fractione, unde in *Canone* 2, generali nulla occurrit quantitas indeterminata præter z, y, & numeros determinatos ad arbitrium sumptos. Et Canonis sic expressi inveniatur fluxio hujusq; fluens in Seriem infinitam resoluta dabit Fractionis $\frac{b}{a+1}$ Logarithmus z; & ex invento z habetur numeri propositi $a+1$ Logarithmus x, nam per *Lem.* 1. $x = l:\overline{a+1} = l:b - z$; & innotescit $l:b$ seu Logarithmus numeri b, quia ex hypothesi is producitur ex multiplicatione numerorum, quorum maximus est minor proposito $a+1$. Adeoq; hæc *Methodus Indirecta* supponit dari Logarithmos omnium Numerorum primorum proposito minorum.

EXEM-

EXEMPLUM IV.

Sumatur $b = a$, unde $z = l: \frac{a}{a + 1}$. Et fit $y = 2a + 1$, *Æquatio* inter y & a cum numeris determinatis ad libitum fumpta; hujus ope exterminetur (a) ex præcedenti erit $z = l: \frac{y - 1}{y + 1}$; cujus fluxio eft $\dot{z} = 2\dot{y} \times \overline{y^2 - 1}|^{-1}$ quæ per Calculum fluentium dat

$$z = -2 \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9} \text{ \&c.} = l: \frac{a}{a+1}.$$

Unde per *Lemma* 1, erit

$$x = l:b + 2 \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9} \text{ \&c.} = l:b - z$$

$= l: a + 1$. Q. E. I.

EXEMPLUM V.

Fiat $b = \sqrt{a^2 + 2a}$: Unde $z = l: \frac{\sqrt{aa + 2a}}{a + 1}$. Et affumatur $y = 2a + 2$; adeoq; erit $z = l: y^{-1}\sqrt{y^2 - 4}$; cujus fluxio eft $\dot{z} = 4\dot{y} \times \overline{y^3 - 4y}|^{-1}$: Unde per Calculum fluentium erit

$$z = -2 \times \frac{1}{y^2} + \frac{\overline{2|}^2}{2y^4} + \frac{\overline{2|}^4}{3y^6} + \frac{\overline{2|}^6}{4y^8} + \frac{\overline{2|}^8}{5y^{10}} \text{ \&c.} = l: \frac{b}{a+1}.$$

Unde per *Lemma* 1,

$$x = l:b + 2 \times \frac{1}{y^2} + \frac{\overline{2|}^2}{2y^4} + \frac{\overline{2|}^4}{3y^6} + \frac{\overline{2|}^6}{4y^8} + \frac{\overline{2|}^8}{5y^{10}} \text{ \&c.} = l: \overline{a + 1}.$$

Q. E. I.

EXEM-

EXEMPLUM VI.

Fiat $b = \sqrt{a^2 + 2a}$; ut in præcedenti *Exemplo*, ſed jam aſſumatur $y^2 = 2a^2 + 4a + 1$. Unde $z = l : \frac{\sqrt{yy-1}}{\sqrt{yy+1}}$ per Canonem generalem 2, cujus fluxio eſt $\dot{z} = 2y\dot{y} \times \overline{y^4 - 1}|^{-1}$; unde per Calculum fluentium erit

$$z = -\frac{1}{y^2} - \frac{1}{3y^6} - \frac{1}{5y^{10}} - \frac{1}{7y^{14}} - \frac{1}{9y^{18}} \text{ \&c.} = l : \frac{b}{a+1}.$$

Unde per *Lemma* 1,

$$x = l : b + \frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} \text{ \&c.} = l : \overline{a+1}.$$

Q. E. I.

In duobus poſtremis *Exemplis* aſſumitur $b^2 = aa + 2a$, & patet quod $aa + 2a$ producitur ex numeris quorum maximus eſt minor quam $a + 1$. Nam $a^2 + 2a = a \times \overline{a+2}$; ſed ex hypotheſi $a + 1$ eſt numerus primus, ergo $a + 2$ eſt numerus par, ergo $\frac{a+2}{2}$ eſt numerus integer; unde $a^2 + 2a = a \times 2 \times \frac{a+2}{2}$, ſed horum quilibet eſt minor quam $a + 1$. Et ex datis Logarithmis partium a, 2, $\frac{a+2}{2}$ habetur Logarithmus producti $a^2 + 2a$ ſeu b^2, adeoq; per *Lem.* 2, Logarithmus numeri b, ſcil. dividendo Logarithmum numeri b^2 per 2.

SCHOLIUM.

Conſtat quod ex utravis Methodo jam explicatâ innumeræ deduci poſſunt Series Logarithmum numeri cujuſvis exhibentes: Quamvis autem variis modis ad arbitrium componi poſſit valor numeri b ex a & determinatis, & quamvis etiam aſſumere liceat quamlibet *Æquationem* inter y & terminos ex a & numeris determinatis compoſitos, ſumma tamen cura laborandum eſt, ut tales ſumantur quantitatum b

(in *Methodo Indirectâ*) & y (in utraq;) valores, qui efficient ut Series Logarithmum exhibentes quam celerrimé convergant. Et postquam Series computatur experientia multas docebit regulas ad facilitandum Calculum Logarithmi ex ista serie computandi. Ut si inveniendus sit Logarithmus numeri cujusvis $n + 1$ per Seriem *Exempli* secundi, ponerem $a + 1 = \frac{n+1}{n}$; unde $a = \frac{1}{n}$, adeoq; $y = \frac{1}{1 + 2n}$ $= \frac{a}{a+2}$. Jam si in ista serie substituatur hic valor numeri y, ejus termini multò celerius convergent, adeoq; minori labore exhibebunt Logarithmum numeri $\frac{n+1}{n}$ ex quo per *Lem.* 1, facilè deducitur Logarithmus quæsitus numeri propositi $n + 1$. Nam $l: \overline{n+1}$ $= l: \frac{n+1}{n} + l: n$.

PROBLEMA I.

Ex dato Logarithmo x *invenire numerum* a + 1, *per Seriem infinitam.*

Postquam invenitur *Æquatio* fluxionalis ex *Æquatione* per regulas in Methodis præcedentibus traditas, tum per Calculum fluentium inveniatur valor quantitatis y; & ex invento y innotescet a, adeoq; & $a + 1$. Q. E. I.

EXEMPLUM.

Si adhibeatur Calculus in *Exemplo* 1 adhibitus, in quo $a = y$, $x = l: \overline{y+1}$ erit $\dot{x} = \frac{\dot{y}}{1+y}$, seu $\dot{x} + y\dot{x} = \dot{y}$; tum (ut fieri solet in hujusmodi casibus per Calculum fluentium) ponatur $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. Et determinando A, B, C, D &c. more solito invenietur $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{2 \times 3}$, $D = \frac{1}{2 \times 3 \times 4}$; Unde erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \frac{x^4}{2 \times 3 \times 4} + \frac{x^5}{2 \times 3 \times 4 \times 5} \text{ \&c.}$$

Et innumeræ hujusmodi Series Numerum ex dato Logarithmo exhibentes facilè sic inveniuntur.

PRO-

PROBLEMA II.

Invenire Seriem, quæ exhibeat Logarithmos Sinuum rectorum.

Sit Radius 1, & Arcûs cujusvis Sinus rectus sit a; unde $\sqrt{1 - aa}$ est Co-sinus, cujus Logarithmus quæsitus sit x; seu $x = l : \sqrt{1 - a^2}$. Tum juxta regulas Methodi directæ assumatur ad libitum quælibet *Æquatio* inter y & a, ex gr. $y = a$; unde $x = \sqrt{1 - y^2}$; cujus fluxio est $\dot{x} = -y\dot{y} \times \overline{1 - yy}|^{-1}$: Unde per Calculum fluentium

$$x = -\tfrac{1}{2} \times y^2 + \tfrac{1}{2} y^4 + \tfrac{1}{3} y^6 + \tfrac{1}{4} y^8 + \tfrac{1}{5} y^{10} \text{ \&c.}$$

Et ex utraq; Methodo, indirecta æque ac directa, innumeræ hujusmodi Series facillimè inveniri possunt.

COROLLARIUM.

Ex Logarithmis Sinuum facile habentur Logarithmi Tangentium. Nam cæteris positis ut in *Prob.* 2, sit t tangens istius Arcûs cujus sinus est a, tum quia $t = \dfrac{a}{\sqrt{1 - aa}}$, erit $l : t = l : \dfrac{a}{\sqrt{1 - aa}}$, sed per *Methodi Indirectæ Lem.* 1, $l : a - l : \sqrt{1 - a^2} = l : \dfrac{a}{\sqrt{1 - aa}}$, ergo $l : t = l : a - l : \sqrt{1 - aa}$; sed ex hypothesi dantur $l : a$ & $l : \sqrt{1 - aa}$, seu Logarithmi Sinûs & Co-sinûs; ergo horum differentia est tangentis Logarithmus quæsitus.

PROBLEMA III.

Invenire Seriem, quæ exhibeat Logarithmos Secantium.

Sit a Tangens, Radius 1, tum $\sqrt{1 + a^2}$ est secans, cujus Logarithmus sit x, seu $x = l : \sqrt{1 + a^2}$; assumatur juxta Methodos præcedentes quævis *Æquatio* inter a & y, ex. gr. $a = y$; unde $x = l : \sqrt{1 + y^2}$, cujus fluxio $\dot{x} = y\dot{y} \times \overline{1 + y^2}|^{-1}$: Unde

$$x = \tfrac{1}{2} y^2 - \tfrac{1}{4} y^4 + \tfrac{1}{6} y^6 - \tfrac{1}{8} y^8 + \tfrac{1}{10} y^{10} - \text{ \&c.}$$

SCHO-

SCHOLIUM.

Eodem modò inveniuntur Series pro Logarithmis *Sinuum Versorum* & Chordarum: Et porro per *Problema* 1, ex dato Logarithmo Sinus, vel Tangentis, vel Secantis invenies Sinum, Tangentem, vel Secantem.

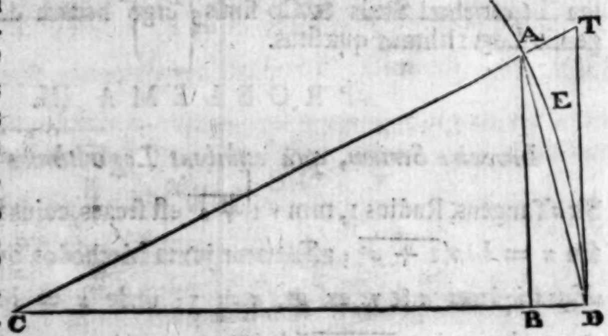
PROBLEMA IV.

Logarithmos per Series inventos transmutare in Logarithmos vulgares.

Dividatur 1.0000000 &c. qui est vulgaris Logarithmus numeri 10 per ejusdem numeri Logarithmum 2.302585 &c. per Series præcedentes inventum, & Quotiens 0.43429481 &c. multiplicatus in cujusvis numeri Logarithmum per Series inventum dabit ejusdem numeri Logarithmum vulgarem. Q. E. I.

SECTIO II.

De Arcu Circulari ejusq; Sinubus, Tangente, & Chorda.



Sit C Centrum & $CD = a$ Radius Arcûs $AED = v$. Sinus rectus $AB = z$, Co-sinus $CB = x$, Sinus versus $BD = y$. Tangens $DT = t$, Secans $CT = s$, & Chorda $AD = c$. In hac Sectione propositum est ostendere, ex dato sinu, vel Tangente vel Chordâ, quomodo inveniatur Arcus, & *e* contrà.

LEMMA

LEMMA.

Invenire relationem inter $\dot{v}$ *&* $\dot{y}$.

Quoniam $\dot{v}^2 = \dot{y}^2 + \dot{z}^2$, ideo si ope *Æquationis* $2ay - y^2 = z^2$ exterminetur $\dot{z}$, erit $\dot{v}^2 = \frac{a^2\dot{y}^2}{2ay - y^2}$; unde $\dot{v} = a\dot{y} \times \overline{2ay - y^2}|^{-\frac{1}{2}}$ Q. E. I.

COROLLARIUM I.

$\dot{v} = a\dot{z} \times \overline{a^2 - z^2}|^{-\frac{1}{2}}$. Ut patebit exterminando y & $\dot{y}$ ex æquatione *Lemmatis* modo inventâ.

COROLLARIUM II.

$\dot{v} = a^2\dot{t} \times \overline{a^2 + t^2}|^{-1}$; Nam $CB, BA :: CD, DT$; i. e. $a - y$, $\sqrt{2ay - y^2} :: a, t$, unde $t = \frac{a\sqrt{2ay - yy}}{a - y}$; cujus ope exterminentur y, $\dot{y}$ ex *Lemmate*, & patebit *Corollarium.*

COROLLARIUM III.

$\dot{v} = 2a\dot{c} \times \overline{4a^2 - c^2}|^{-\frac{1}{2}}$. Ut patebit si ex *Lemmate* exterminentur y & $\dot{y}$, ope hujus $2ay = cc$.

PROBLEMA I.

Ex dato Sinu Verso y, *invenire Arcum* v.

Quia per *Lemma* præcedens $\dot{v} = a\dot{y} \times \overline{2ay - y^2}|^{-\frac{1}{2}}$, ideo si $\overline{2ay - y^2}|^{-\frac{1}{2}}$ resolvatur in Seriem, per Calculum fluentium invenietur $v = \sqrt{2ay} \times 1 + \frac{y}{2 \times 3 \times 2a} + \frac{3y^2}{2 \times 4 \times 5 \times 4a^2}$ &c.

PROBLEMA II.

Ex dato Sinu recto invenire Arcum.

Quia $\dot{v} = a\dot{z} \times \overline{aa - zz}|^{-\frac{1}{2}}$, ideo si $\overline{aa - zz}|^{-\frac{1}{2}}$ resolvatur in Seriem invenietur per Calculum fluentium

$$v = z + \frac{z^3}{2 \times 3aa} + \frac{3z^5}{2 \times 4 \times 5a^4} + \frac{3 \times 5z^7}{2 \times 4 \times 6 \times 7a^6} \text{ \&c.}$$

PROBLEMA III.

Ex data Tangente invenire Arcum.

Per *Coroll.* 2, $\dot{v} = a^2\dot{t} \times \overline{aa+tt}|^{-1}$; resolvatur itaq; $\overline{a^2+t^2}|^{-1}$ in Seriem infinitam, & per Calculum fluentium invenietur

$$v = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8} - \frac{t^{11}}{11a^{10}} \text{ \&c.}$$

PROBLEMA IV.

Ex data Chorda invenire Arcum.

Quia per *Coroll.* 3. $\dot{v} = 2a\dot{c} \times \overline{4a^2 - c^2}|^{-\frac{1}{2}}$, ideo resolvatur $\overline{4a^2 - c^2}|^{-\frac{1}{2}}$ in Seriem infinitam, & per Calculum fluentium invenietur

$$v = c + \frac{c^3}{\overline{2}|^3 \times 3 \times aa} + \frac{3c^5}{\overline{2}|^5 \times 4 \times 5 \times a^4} + \frac{3:5:c^7}{\overline{2}|^7 \times 4 \times 6 \times 7 \times a^6} \text{ \&c.}$$

SCHOLIUM.

Eodem modo ex dato Co-finu x vel Secante s inveniri potest Arcus v.

COROLLARIUM.

Si Arcus v sit perparvus, tum Sinus rectus, Tangens, Chorda & Arcus sunt quamproximé æquales. Nam in hoc casu omnes Serierum termini præter primum sunt magnitudinis contemnendæ, & proinde $v = z$ per *Prob.* 2, & $v = t$ per *Prob.* 3; & $v = c$ per *Prob.* 4.

PROBLEMA V.

Ex dato Arcu invenire Tangentem.

Quia per *Coroll.* 2. $\dot{v} \times \overline{aa+tt} = aa\dot{t}$; ideo per Regulas Calculi fluentium assumatur $t = Av + Bv^2 + Cv^3 + Dv^4 + Ev^5$ &c. Et determinando A, B, C &c. more solito invenies $A = 1$, $B = 0$, $C = \frac{1}{3aa}$, $D = 0$, $E = \frac{2}{3 \times 5a^4}$, &c. Unde $t = v + \frac{v^3}{3aa} + \frac{2v^5}{3 \times 5a^4}$ &c. Q. E. I.

SCHO-

SCHOLIUM.

Similiter, ex dato Arcu invenies Sinum, Co-finum, *Sinum Versum* Tangentem & Chordam. Sic fi inveniendus esset Sinus rectus z; Quia per *Coroll.* 1, $\dot{v}\sqrt{a^2 - z^2} = a\dot{z}$, ideo per Regulas Calculi fluentium assumo $z = Av + Bv^2 + Cv^3 + Dv^4 + Ev^5$ &c. Et determinando A, B, C &c. invenio $A = 1$, $B = 0$, $C = \frac{-1}{2 \times 3\, a^2}$, $D = 0$, $E = \frac{1}{2 \times 3 \times 4 \times 5\, a^4}$: Unde erit $z = v - \frac{v^3}{2 \times 3\, a^2} + \frac{v^5}{2 \times 3 \times 4 \times 5\, a^4}$ &c.

COROLLARIUM.

Ex his deducitur Methodus facilis & genuina computandi Tabulas Sinuum, Tangentium, Secantium, *&c.* quam paucis explicare visum est; sunt enim in Geometria practica plurimi momenti.

1. Sit Radius $a = 1.00000$ &c. tum totius Circuli Circumferentia erit 6.2831853 &c. $= p$.

2. Invenire rationem inter Radium & arcum quemvis per gradus vel minuta designatum. Ut tota Circumferentia per minuta prima designata *(i. e.)* ut 21600 ad arcum quemvis per minuta prima designatum, sic 6.2831853, ad arcum per partes radii designatum. *Ex. gr.* invenienda sit ratio inter radium & Arcum unius minuti primi; proportio erit $21600 : 1' :: 6 : 2831853$, ad quartum scil. .0002908; adeoq; radius est ad arcum $1'$ ut 1. 00000 ad .0002908.

3. Invenire Sinum rectum Arcus unius minuti; quoniam $a = 1$. Et $v = .0002908$, ideo $z = .0002908 - \frac{\overline{.0002908}|^3}{6} + \frac{\overline{.0002908}|^5}{120}$ &c. per Seriem in *Schol. Problematis* 5. Eodem modo invenies Sinum rectum cujusvis Arcus; & sic computabitur Tabula Sinuum rectorum.

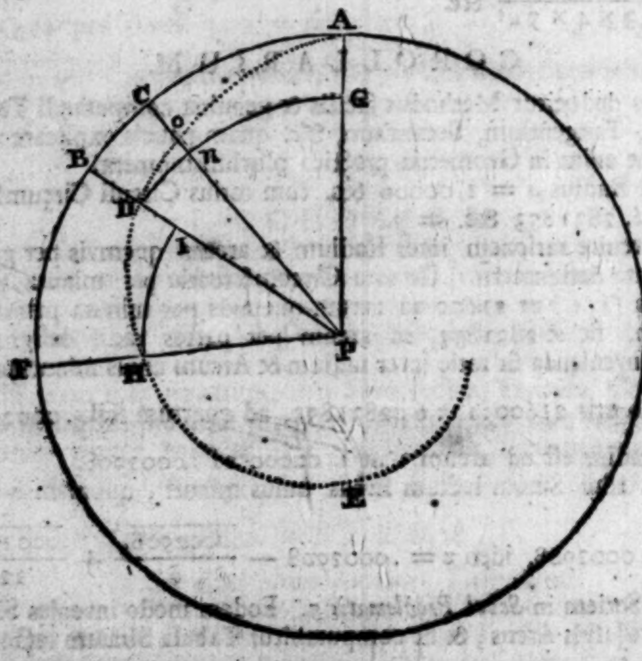
Et si in Serie *Prob.* 5, pro v substituatur .0002908, erit Arcûs $1'$ Tangens $t = .0002908 + \frac{\overline{.0002908}|^3}{3}$ &c. Et sic computabitur Tabula Tangentium. Facilius tamen ex computatis sinubus rectis invenientur Tangentes, Secantes, &c. ut notum est.

SECTIO.

SECTIO III.

De Curva Loxodromica.

LOxodromia (quæ & linea Rhombica dicitur) est Spiralis in superficie Globi descripta, quam Meridiani secant angulo constanti. Est nimirum linea illa, quam Navis cursu suo describit, quando ita ad acûs Magneticæ situm se componit, ut Carina ejus Angulum aliquem constantem cum omnibus (quos secat) Meridianis comprehendat.



Sit *ACBF* Æquator, ejus polus *P*, Meridiani *PA*, *PC*, *PB*, secantes Curvam *AoDE* angulo constanti *PoD*, tum Curva illa *AoDE* est *Loxodromios* seu Rhombica, & Angulus ille constans *PoD* vocatur Angulus Rhombi. Sit jam *Do Loxodromiæ* pars infinitè parva, & centro *P*, radio *PD* describatur Arcus *DG* Meridianos *PDB*, *PoC*, & *PA* secans in punctis *D*, *n*, *G*, tum Triangulum

lum Don (quamvis in Superficie Sphæræ descriptum) à plano non differt; nam sicut (per Calculi fluxionum principia) pars Curvæ cujusvis infinite parva a recta nôn differt, sic Superficiei cujuscunq; rotundæ pars infinite parva Don tanquam plana habetur. Trianguli Dno (quod *Loxodromicum* vocabitur) unusquisq; Angulus est constans. Nam Dno est rectus, quia Dn est arcus centro P & radio Pn descriptus, & noD est constans ex Natura *Loxodromiæ*; ergo etiam ejus complementum scil. Angulus nDo est constans. Deniq; Do est *Loxodromiæ* ADE, on Meridiani (seu Latitudinis) Dn paralleli per Locum D transeuntis, & BC æquatoris (seu Longitudinis) quantitas fluxionalis.

PROBLEMA I.

Ex datis duorum Locorum A, D Latitudinibus, invenire *Loxodromiæ* Longitudinem inter illa duo Loca interceptam.

Sit Sphæræ radius $PA = a$, Latitudinum datarum differentia $AG = y$, Longitudinum differentia $AB = x$, r Sinus Latitudinis Loci D, ejusq; Co-sinus z; $AoD = v$. Anguli Rhombi noD Sinus sit b, Secans e, & Tangens m. Anguli verò nDo Sinus c, Secans s, & Tangens t; quæ sex postremæ lineæ sunt constantes, ut patet ex dictis.

Jam in Triangulo *Loxodromico* si on sit radius, tum $oD = e$; unde per *Trigonometriam* $a : on :: e : oD$, i. e. $a : \dot{y} :: e : \dot{v}$, seu $a\dot{v} = e\dot{y}$; unde regrediendo ad fluentes $av = ey$, seu $v = \frac{ey}{a}$. Q. E. I.

Aliter. Si Do sit radius, tum on erit Sinus Anguli nDo seu c; unde per *Trigonometriam* $c : on :: a : Do$, i. e. $c : \dot{y} :: a : \dot{v}$ seu $c\dot{v} = a\dot{y}$; unde regrediendo ad fluentes $cv = ay$, seu $v = \frac{ay}{c}$.

COROLLARIUM I.

Ex data itineris Longitudine $AD = v$, invenietur Latitudinum differentia $AG = y = \frac{cv}{a}$.

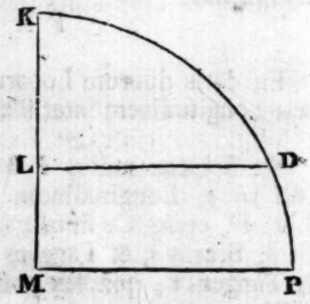
Q COROL-

COROLLARIUM II.

Si duorum D, H, Latitudinum differentia DI fit æqualis aliorum quorumlibet Locorum D, A, Latitudinum differentiæ AG; erit *Loxodromiæ* (per hæc quatuor loca transeuntis) pars inter A & D æqualis parti inter D & H interceptæ. Nam per Solutionem problematis posteriorem

$AD = \frac{a}{c} \times AG$ & $DH = \frac{a}{c} \times DI$; sed ex hypothesi $AG = DI$; ergo $AD = DH$. Q. E. I.

LEMMATA.



1. Ut radius ad Co-sinum Latitudinis loci D, sic æquatoris fluxio $\dot{BC}$ ad fluxionem Circuli paralleli per locum D transeuntis, *i. e.* $a : z :: \dot{x} : \dot{nD}$: Seu $\dot{nD} = \frac{z\dot{x}}{a}$. Nam a est radius æquatoris, & z est radius paralleli; & BC, Dn sunt suorum Circulorum arcus similes.

2. Invenire *Æquationem*, quæ exprimet relationem inter $\dot{y}$ & $\dot{r}$.

Sit M Centrum Meridiani PDK per locum D transeuntis, P polus æquatoris, PDK Quadrans, adeoq; K est punctum in æquatore; & proinde KD est Latitudo loci D, cujus Sinus sit DL, adeoq; ML est Co-sinus Latitudinis. Jam denotetur arcus $KD = y$ (sicut in præcedenti Figura $AG = y$, ubi Meridiani projiciuntur in rectas;) sed jam designantur $DL = r$, & $ML = z$; & ex Circuli Natura $a^2 - r^2 = z^2$, & ex principiis Calculi fluxionum $\dot{y} = \sqrt{\dot{r}^2 + \dot{z}^2}$. Sed $\dot{z}^2 = \frac{r^2\dot{r}^2}{a^2 - r^2}$; ergo $\dot{y} = \frac{a\dot{r}}{z}$. Q. E. I.

3. Invenire *Æquationem*, quæ exprimet relationem inter $\dot{y}$ & $\dot{x}$. In Triangulo Dno si Dn sit radius, tum on erit tangens Anguli nDo seu t, adeoq; $a : Dn :: t : on$; sed $on = \dot{y}$, & $Dn = \frac{z\dot{x}}{a}$ (per *Lem.* 1.) ergo $a : \frac{z\dot{x}}{a} :: t : \dot{y}$, seu $a\dot{y} = \frac{zt\dot{x}}{a}$. Q. E. I.

Aliter.

Aliter. Sit on Radius, & tum Dn erit Tangens Anguli Rhombi noD seu m: Unde $a : on :: m : Dn$, i.e. $a : \dot{y} :: m : \frac{z\dot{x}}{a}$; Seu $m\dot{y} = z\dot{x}$. Q. E. I.

COROLLARIUM.

$z^2\dot{x} = am\dot{r}$. Nam $\dot{y} = \frac{a\dot{r}}{z}$ (per *Lem.* 2) & $\dot{y} = \frac{z\dot{x}}{m}$ (per *Lem.* 3)

Ergo $\frac{z\dot{x}}{m} = \frac{a\dot{r}}{z}$; Unde $z^2\dot{x} = am\dot{r}$. Q. E. D.

PROBLEMA II.

Ex data Loci Latitudine invenire ejus Longitudinem.

Sit Locus D, ejus Latitudinis datæ Sinus r, Co-finus z, Anguli Rhombi Tangens fit m, PA primus Meridianus, Longitudo quæsita $AB = x$.

Jam (per *Lem.* 3. *Coroll.*) $\dot{x} = \frac{am\dot{r}}{zz}$; Sed $z^2 = a^2 - r^2$; ergo $\dot{x} = \frac{am\dot{r}}{a^2 - r^2}$; Unde per Calculum fluentium invenietur.

$$x = m \times \frac{r}{a} + \frac{r^3}{3a^3} + \frac{r^5}{5a^5} + \frac{r^7}{7a^7} + \frac{r^9}{9a^9} + \frac{r^{11}}{11a^{11}} \text{ \&c. Q. E. I.}$$

COROLLARIUM.

Hinc habetur differentia Longitudinum BF duorum quorumvis Locorum D, H, per quæ tranfit eadem *Loxodromia* ADH.

Sit n Sinus Latitudinis Loci alterius H, cæteris defignatis ut in *Prob.* 2. Jam per Seriem in problemate exhibitam erit

$$ABF = m \times \frac{n}{a} + \frac{n^3}{3a^3} + \frac{n^5}{5a^5} + \frac{n^7}{7a^7} \text{ \&c.}$$

$$AB = m \times \frac{r}{a} + \frac{r^3}{3a^3} + \frac{r^5}{5a^5} + \frac{r^7}{7a^7} \text{ \&c.}$$

$$\text{Unde } BF = m \times \frac{n-r}{a} + \frac{n^3 - r^3}{3a^3} + \frac{n^5 - r^5}{5a^5} + \frac{n^7 - r^7}{7a^7} \text{ \&c.}$$

Q. E. I.

SECTIO

SECTIO IV.

De Transmutatione Figurarum Curvilinearum.

Figura quævis Curvilinea proposita in aliam transmutari dicitur, quando hujus pars assignari potest, quæ æqualis sit portioni cuilibet Figuræ propositæ. Et quamvis vulgare sit apud *Geometras* problema Figuram quamlibet propositam in alias innumeras transmutare; nullus tamen hactenùs ostendit Methodum determinandi quænam earum omnium sit simplicissima, quæ cum propositâ comparari possit. Hujusmodi autem Methodus egregium foret incrementum illius *Geometriæ* partis, quæ spectat Figurarum Curvilinearum Quadraturas. Sperandum itaq; aliquem eandem aliquando aggressurum, cui Deus ingenium & otium problematis difficultati æqualia est largitus. In tractatu meo de Quadraturis Anno 1693, edito Specimena aliquot exhibui comparandi Figuras propositas cum ejusdem Ordinis simplicissimis. Sed hæc omnia nihil prorsus sunt, respectu Methodi generalis adhuc desideratæ. Quandò hujusmodi Speculationibus Geometricis (multos ante annos) incumberem, occurrit mihi *Theorema* satis (ut opinor) elegans, cujus ope innumeræ Figuræ cum Figuris generis Parabolici comparari, adeoq; earum Quadraturæ facilè determinari possunt. *Æquatio* autem innumeras illas Curvas definiens est $y = bx^{c} \times \overline{a + nx^{e}}|^{r}$. Quarum Quadratura generalis per Seriem infinitam dedit Illustrissimus *Newtonus*. Hujus *Theorematis* investigationem Analyticam in hac Sectione exhibeo; quoniam ad Curvas magis compositas eadem cum successu adhiberi potest.

Methodi autem transmutandi Figuras Curvilineas præstantia in hoc etiam non parùm elucescit, quod illius ope Figuras magis compositas cum simplicioribus sæpe numerò comparari possumus, & Tabulas construere innumerarum Figurarum, quæ cum Circulo, Ellipsi, Hyperbolâ, &c. comparari possunt; quæ usus habere eximios in determinandis Figurarum propositarum Quadraturis, quando Figura proposita in his Tabulis continetur. Hujusmodi aliquot egregias exhibuit Dignissimus *Newtonus*, incomparabilis harum Scientiarum Promotor, in Tractatu suo de Quadraturis, qui Libro de Optica subnectitur.

In

In Tractatu nostro Geometrico de Quadraturis An. 1693 edito usus non contemnendos invenies pro transmutatione Spatiorum Curvilineorum, quos ex *Theoremate* quodam *Baroviano* deduxi. Et quidem præclarissimi Viri Lectiones Geometricæ tot interioris *Geometriæ* propositionibus replentur, ut non pauca magniq; momenti ex iis deducere possit, qui operam sedulo navare vellet: Sed, his omissis, Figurarum Curvilinearum transmutationes aggrediar, postquam præmisi hoc quod sequitur.

LEMMA I.

Figuras quotcunq; invenire quarum Areæ sint datæ cuilibet Figuræ æquales.

Sint Figuræ datæ Abscissa x & Ordinata y; & Figuræ quæsitæ Abscissa z & Ordinata s: Unde ex conditione *Problematis* propositi erit $F : y\dot{x} = F : s\dot{z}$: Adeoq; $y\dot{x} = s\dot{z}$. Assumatur quælibet *Æquatio* inter terminos ex z, x & quantitates quasvis determinatas utcunq; compositas. Ope hujus assumptæ & *Æquationis* datam Figuram definientis exterminentur y, x & $\dot{x}$ ex *Æquatione* præcedente $y\dot{x} = s\dot{z}$, & habebitur *Æquatio* exprimens relationem inter s & z, quæ Curvam quæsitam definiet: Erit utiq; Area Figuræ sic inventæ æqualis Areæ Figuræ datæ, dummodo talis sit utriusq; Abscissarum z, x relatio, qualis per *Æquationem* assumptam exprimitur: Et quia *Æquatio* assumpta modis innumeris variari potest, ideo innumeræ inveniri possunt Figuræ, quæ sint datæ cuilibet Figuræ æquales. *Q. E. I.*

EXEMPLUM.

Sit $y = \sqrt{aa - xx}$ *Æquatio* Circulum definiens. Assumatur ad libitum $ax = zz$: Unde $\dot{x} = \frac{2z\dot{z}}{a}$, & $y = \sqrt{aa - \frac{z^4}{aa}} = \frac{1}{a}\sqrt{a^4 - z^4}$, ergo si in *Æquatione* $y\dot{x} = s\dot{z}$ pro y & $\dot{x}$ substituantur eorum modo inventi valores, erit $\frac{2\dot{z}}{aa}\sqrt{a^4 - z^4} = s\dot{z}$; unde $\frac{2z}{aa}\sqrt{a^4 - z^4} = s$, quæ definit Curvam, cujus Area est æqualis Areæ circulari, dummodo abscissarum z, x relatio sit illa, quæ per $ax = zz$ exprimitur.

PROBLEMA I.

Omnes Figuras Binomiales Geometricè quadrabiles in Parabolicas transmutare.

Sit $y = bx^c \times \overline{a + nx^e}|^r$ *Æquatio* definiens omnes Curvas Binomiales; sitq; *Æquatio* assumpta $z = \overline{a + nx^e}|^r$: Ex his duabus per *Lemma* præcedens invenietur

$$s = \frac{bz^{\frac{1}{r}} \times \overline{z^{\frac{1}{r}} - a}|^{\frac{c+1}{e} - 1}}{re \times \overline{n}|^{\frac{c+1}{e}}}$$

Patet quod si $\frac{c+1}{e}$ sit numerus integer ac positivus, tum valor Ordinatæ s constabit terminis simplicibus ex z & determinatis compositis; & proinde *Æquatio* modo inventa semper definiet Curvam Parabolicam, quando $\frac{c+1}{e}$ est numerus integer & positivus, *i. e.* quando Figuræ Binomiales sunt Geometricè quadrabiles.

COROLLARIUM.

Figuræ Binomiales Geometricè quadrabilis Quadraturam determinare.

Reducatur *Æquatio* Curvam datam definiens ad formam *Æquationis* generalis in *Prob.* 1. exhibitæ; & ex debita hujus cum illa comparatione invenientur valores quantitatum b, c, a, n, e, r; qui substituti in *Æquatione* per *Prob.* 1. inventa dabunt *Æquationem* quæ Figuram definiet parabolicam; cujus inveniatur quadratura per Methodos notissimas; in qua si pro z substituatur ejus valor $\overline{a + nx^e}|^r$ habebitur Quadratura Figuræ propositæ. *Q. E. I.*

EXEM-

EXEMPLUM.

Sit $y = x \times \overline{mm + xx}|^{\frac{1}{2}}$ *Æquatio* definiens Curvam datam, ex comparatione hujus cum generali erit $b = 1$, $c = 1$, $a = mm$: Et $n = 1$, $e = 2$, $r = \frac{1}{2}$. Unde $s = z^2$; adeoq; Area ejus $F : s\dot{z}$ $= \frac{z^3}{3}$. Sed ex assumpta $z = \overline{mm + xx}|^{\frac{1}{2}}$, adeoq; $\frac{1}{3}z^3 = \frac{\overline{m^2 + x^2}|^{\frac{3}{2}}}{3}$ $= F : s\dot{z}$: Sed ex Hypothesi $F : y\dot{x} = F : s\dot{z}$; Ergo $F : y\dot{x}$ $= \frac{\overline{mm + xx}|^{\frac{3}{2}}}{3}$. Q. E. I.

SCHOLIUM.

Quamvis *Lemma* præcedens usus habeat maximos non modo in determinandis quadraturis figurarum Geometricè quadrabilium, sed etiam in transmutanda figura quavis non quadrabili in simpliciorem; unde formari possunt Tabulæ innumerarum Figurarum, quarum Quadraturæ dependent a Circuli, Ellipseos, Hyperbolæ, aliarumve figurarum simpliciorum quadraturis: Sed quia datur nulla Regula certa assumendi *Æquationem* inter z & x, per quam debita fiat figuræ propositæ transmutatio; itaq; si figurarum Curvilinearum Quadraturas ad parabolicarum Quadraturas reducere velimus, ad alias Methodos recurrere necesse erit. Et quidem una hujusmodi infra tradetur, adeò generalis ut figuræ cujusvis Quadratura per parabolicas quam proximè exhiberi possit.

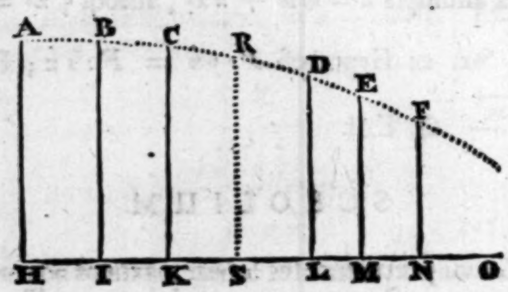
LEMMA II.

Lineam Curvam generis parabolici describere, quæ per assignata quotcunq; puncta A, B, C, D, E, F, &c. *transibit.*

A punctis assignatis ad rectam HO utcunq; ductam demittantur perpendiculares AH, BI, CK, DL, EM, FN, &c. Sit $HS = y$ quævis Abscissa, ejusq; Ordinata $SR = z$. Abscissæ verò datæ sint $HI = b$, $HK = k$, $HL = l$, $HM = m$, $HN = n$; earumq; Ordinatæ datæ sint $AH = a$, $BI = b$, $CK = c$, $DL = d$, $EM = e$, $FN = f$.

Jam

Jam ex natura Curvæ parabolicæ, *Æquatio* ipsam definiens erit $a + By + Cy^2 + Dy^3 + Ey^4 + Fy^5$ &c. $= z$; ubi B, C, D, E, F sunt coefficientes incognitæ in hunc modum inveniendæ. Pro y substituantur successive Abscissæ datæ h, k, l, m, n; & pro z earum Ordinatæ datæ b, c, d, e, f: Unde tot prodibunt *Æquationes*, quot sunt puncta assignata; scil.



$$
\begin{aligned}
a + hB + h^2C + h^3D + h^4E + h^5F &= b,\\
a + kB + k^2C + k^3D + k^4E + k^5F &= c,\\
a + lB + l^2C + l^3D + l^4E + l^5F &= d,\\
a + mB + m^2C + m^3D + m^4E + m^5F &= e,\\
a + nB + n^2C + n^3D + n^4E + n^5F &= f.
\end{aligned}
$$

Reductione harum *Æquationum* invenies quantitates B, C, D, E, F &c. adeoq; habetur *Æquatio* Curvam parabolicam quæsitam definiens. Q. E. I.

SCHOLIUM.

Ut pateat progressio harum coefficientium B, C, D &c. in infinitum, per communis *Algebra* ductum invenies sequentes faciendas esse positiones: In quibus verò notandum quod literarum p, q, r, s &c. exponentes non denotent harum quantitatum potestates Algebraicas; sed quod diversarum literarum loca supplentes terminorum progressioni detegendæ inserviant.

A B

$\frac{b-a}{b}=p$	$\frac{p^2-p}{k-b}=p^u$	$\frac{b^2-k^2}{k-b}=q$	$\frac{b^3-k^3}{k-b}=r$	$\frac{b^4-k^4}{k-b}=s$	
$\frac{c-a}{k}=p^2$	$\frac{p^3-p}{l-b}=p^{2u}$	$\frac{b^2-l^2}{l-b}=q^2$	$\frac{b^3-l^3}{l-b}=r^2$	$\frac{b^4-l^4}{l-b}=s^2$	
$\frac{d-a}{l}=p^3$	$\frac{p^4-p}{m-b}=p^{3u}$	$\frac{b^2-m^2}{m-b}=q^3$	$\frac{b^3-m^3}{m-b}=r^3$	$\frac{b^4-m^4}{m-b}=s^3$	&c.
$\frac{e-a}{m}=p^4$	$\frac{p^5-p}{n-b}=p^{4u}$	$\frac{b^2-n^2}{n-b}=q^4$	$\frac{b^4-n^4}{n-b}=r^4$	$\frac{b^4-n^4}{n-b}=s^4$	
$\frac{f-a}{n}=p^5$	&c.	&c.	&c.	&c.	
&c.					

C D

$\frac{p^{2u}-p^u}{q-q^2}=p^{uu}$	$\frac{r^2-r}{q-q^2}=q^u$	$\frac{s^2-s}{q-q^2}=r^u$	
$\frac{p^{3u}-p^u}{q-q^3}=p^{2uu}$	$\frac{r^3-r}{q-q^3}=q^{2u}$	$\frac{s^3-s}{q-q^3}=r^{2u}$	&c.
$\frac{p^{4u}-p^u}{q-q^4}=p^{3uu}$	$\frac{r^4-r}{q-q^4}=q^{3u}$	$\frac{s^4-s}{q-q^4}=r^{3u}$	
&c.	&c.	&c.	

E F

$\frac{p^{2uu}-p^{uu}}{q^u-q^{2u}}=p^{uuu}$	$\frac{r^{2u}-r^u}{q^u-q^{2u}}=q^{uu}$	
		&c.
$\frac{p^{3uu}-p^{uu}}{q^u-q^{3u}}=p^{2uuu}$	$\frac{r^{3u}-r^u}{q^u-q^{3u}}=q^{2uu}$	
&c.	&c.	

G H

$$\frac{p^{2uuu}-p^{uuu}}{q^{uu}-q^{2uu}}=p^{u4} \text{ \&c.}$$

&c.

Ex his paucis manifesta est positionum harum tam horizontalium quam verticalium progressio in infinitum. Et proinde patet etiam progressio coefficientium B, C, D &c. scil.

$$F = p^{uuuu}$$
$$E = p^{uuu} + q^{uu} F$$
$$D = p^{uu} + q^u E + r^u F$$
$$C = p^u + qD + rE + sF$$
$$B = p - bC - b^2 D - b^3 E - b^4 F$$

Elegantem *Problematis* hujus Solutionem exhibuit Illustrissimus *Newtonus* pag. 446. *Princip. Math. Phil. Naturalis*: Sed quia & Analysin & Demonstrationem ipsi visum fuit omittere; hanc qualemcunq; breviter exponere non prorsus inutile judicavi.

PROBLEMA II.

Figuræ cujusvis datæ Quadraturam invenire.

Sit *HAF* figura proposita, per *Lem.* præcedens inveniatur *Æquatio* definiens Curvam parabolicam, quæ transibit per Curvæ hujus puncta quotlibet *A*, *B*, *C*, *R*, *F*; eritq; Curvæ hujus parabolicæ Quadratura per Methodos vulgares invenienda figuræ propositæ æqualis. Q. E. I.

SCHOLIUM.

Quo plura sunt puncta Curvæ propositæ *A*, *B*, *C*, *R*, *F* per quæ transit Curva parabolica, eò propius accedit hujus ad Aream illius.

Notandum. Quod hisce subjungere decreveram Calculum inveniendi *Theoremata* mea pro locis Geometricis, quæ cum tractatu nostro de Quadraturis edita sunt An. 1693. Sed me Labore illo exoneravit Illustrissimus Hospitalius, qui operis sui posthumi librum scripsit integrum 7, de *Theorematum* horum Calculo & diffusâ Methodi nostræ explicatione. Multumq; gratulor nostra qualiacunq; tantoperè in *Gallia* probari, ut ipse Hospitalius (magnum Scientiarum harum Decus & promotor egregius) eadem transcribere, ejusq; operis posthumi Editor eadem luci denuo exponere dignati fuerint.

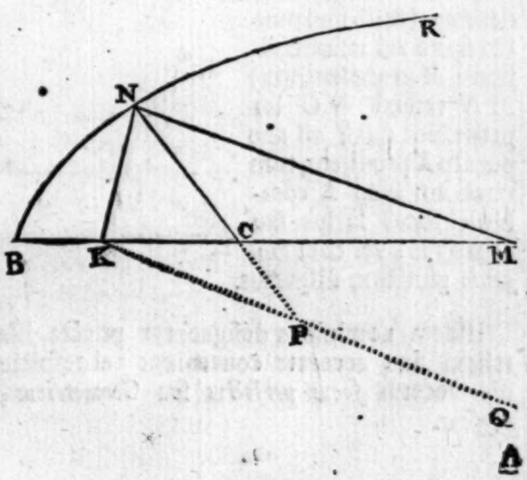
OPTICA

OPTICA ANALYTICA.

DE
CATOPTRICA
LIBER PRIMUS.

LEMMA.

SIT *BNR* Linea Curva quæcunq; ejus Axis *MB*, punctum radians *M*, Radius incidens *MN*, reflexus *NK* axem *BM* secans in *K*, sitq; *NC* normalis ad Curvam & axem secans in *C*. Erit rectangulum sub recta *CK* & Radio incidenti *MN* æquale rectangulo sub recta *CM* & radio reflexo *NK*.

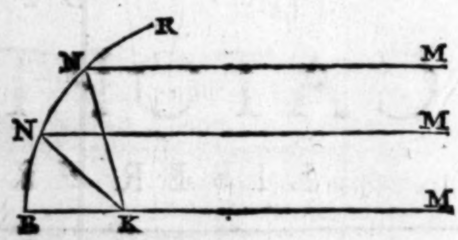


A puncto K ducatur KQ ad MN parallela, & producatur NC donec lineæ KQ occurrat in puncto P. Jam quia NM, QK sunt parallelæ, ideo triangula CKP, CMN sunt similia; ergo $CK : PK :: CM : NM$: Unde $CK \times NM = PK \times CM$. Sed per Legem *Catoptricæ* fundamentalem anguli MNP, KNP sunt æquales & ob parallelas inde NM, KQ æquales etiam sunt anguli MNP, KPN; ergo $KNP = KPN$, & proinde $PK = KN$; adeoq; si in *Æquatione* præcedente substituatur KN pro PK, erit $CK \times NM = KN \times CM$. Q. E. D.

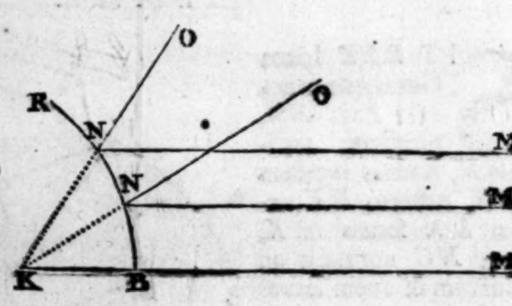
COROLLARIUM.

Si punctum radians M sit infinitè distans, *i. e.* si Radii incidentes MN sint physicè paralleli, tum erit $CK = KN$; nam in hoc casu $NM = CM =$ infinito.

Definitiones.



Quando omnes Radii procedentes a puncto M finitè vel infinitè distanti ita reflectuntur a Curva reflectente BNR, ut omnes reflexi NK in uno puncto K conveniant; tum punctum illud K vocabitur focus Realis seu affirmativus, vel cum *Hugenio* punctum concursûs.



Quando omnium incidentium (& ab uno puncto finitè vel infinitè distanti M procedentium) MN reflexi NO ita procedunt quasi ab uno puncto K prodirent, tum punctum illud K vocabitur focus falsus seu negativus; vel cum *Hugenio* punctum dispersus.

Iisdem nominibus designentur puncta illa, ad quæ tamen omnes reflexi non accurate conveniunt vel respiciunt. Distinctionis gratiâ ille vocetur *focus perfectus* seu *Geometricus*; hic verò *imperfectus* seu *physicus*.

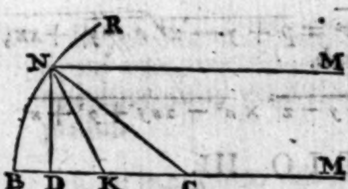
In

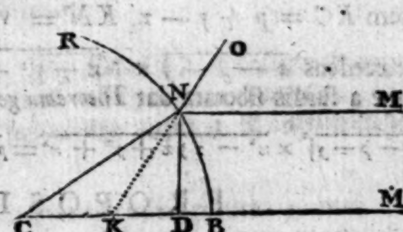
In investigatione *Foci Physici*, punctum illud, ultra vel citra quod nullus Reflexus axem secat, *Focum* quæsitum esse intelligo. Eadem etiam nomina tribuo punctis illis, in quibus Radii refracti colliguntur, vel ad quæ dispersi respiciunt.

Catoptricæ & *Dioptricæ* duæ sunt partes, *Directa* & *Inversa*; per partem *Directam* intelligo illam, per quam, ex data superficie inflectente, inveniatur *Focus*. Pars *Opticæ Inversa* est illa, per quam, ex datis *Foco* & puncto Radiante, inveniatur superficies inflectens, pro qua ubiq; substituto Lineam illam Curvam, cujus rotatione circa axem generatur Superficies ipsa inflectens.

PROPOSITIO I.

In qua traditur Theorema generale *pro Radiis parallelis in quamlibet superficiem concavam vel convexam incidentibus.*

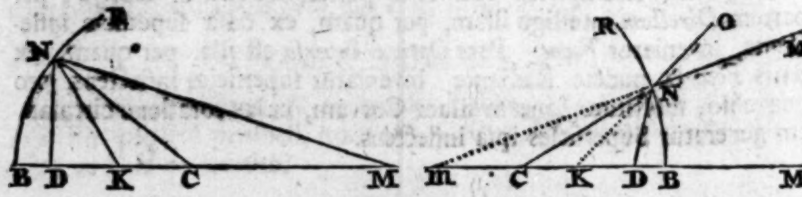




Sit BM axis Curvæ reflectentis BNR, cui parallelus sit Radius incidens MN, cujus reflexus sit NK, quando Radii incidunt in Curvam concavam, ut in *Fig.* 1. vel KNO, quando in convexam, ut in *Fig.* 2. A puncto incidentiæ N ducantur ND ad BM, & NC ad Curvam BNR normales; sitq; Abscissa $BD = y$, Ordinata $DN = x$, subnormalis $DC = p$, & $BK = z$. Unde $KC = p + y - z$: Sed $KN = \sqrt{DK^2 + DN^2} = \sqrt{\overline{z-y}|^2 + x^2}$: Unde per *Lemmatis* præcedentis *Corollarium* $p + y - z = \sqrt{\overline{z-y}|^2 + x^2}$, quæ à surdis liberata dabit *Theorema generale* $p^2 + 2py - 2pz = x^2$.

PROPOSITIO II.

In qua traditur Theorema generale *pro Radiis in superficiem quamlibet concavam divergentibus vel in convexam convergentibus.*

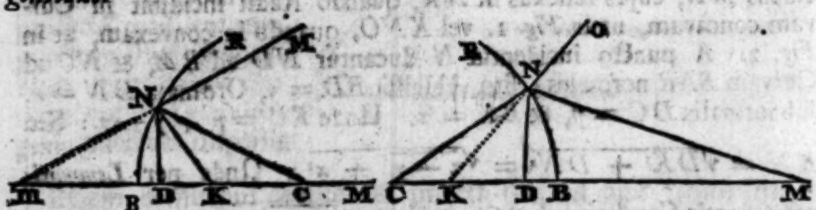


Cæteris designatis ut in præcedenti, sit $Bm = a$ in *Fig.* 1. vel $B'm = a$ in *Fig.* 2, ubi m est punctum Convergentiæ radiorum incidentium MN; erit $CM = a - p - y$ (Fig. 1.) vel $Cm = a - p - y$ (Fig. 2.) & $MN = \sqrt{\overline{a - y}|^2 + xx}$; vel $mN = \sqrt{\overline{a - y}|^2 + xx}$; item $KC = p + y - z$, $KN = \sqrt{\overline{z - y}|^2 + x^2}$; unde per *Lemma* præcedens $\overline{a - p - y} \times \sqrt{\overline{z - y}|^2 + x^2} = \overline{p + y - z}\sqrt{\overline{a - y}|^2 + xx}$; quæ a surdis liberata dat *Theorema generale.*

$$\overline{a - p - y}|^2 \times \overline{z^2 - 2yz + y^2 + x^2} = \overline{p + y - z}|^2 \times \overline{a^2 - 2ay + y^2 + x^2}.$$

PROPOSITIO III.

In qua traditur Theorema generale *pro Radiis in quamlibet superficiem concavam convergentibus, vel in convexam divergentibus.*



Omnibus ut in præcedenti designatis, erit Cm (vel CM in *Fig.* 2.) $= a + p + y$; MN vel $mN = \sqrt{\overline{a + y}|^2 + xx}$, $CK = p + y - z$, &

& $KN = \sqrt{\overline{z-y}|^2 + xx}$; unde per *Lemma* præcedens invenietur $\overline{a+p+y}\sqrt{\overline{z-y}|^2 + x^2} = \overline{p+y-z}\sqrt{\overline{a+y}|^2 + x^2}$, quæ a surdis liberata dat hocce *Theorema generale.*

$$\overline{a+p+y}|^2 \times \overline{z^2 - 2yz + y^2 + x^2} = \overline{p+y-z}|^2 \times \overline{a^2 + 2ay + y^2 + x^2}.$$

Ex his tribus propositionibus, quæ ad *Catoptricam* spectant, omnia.

REGULÆ.

Quæ in Theorematum *præcedentium usu sunt observanda.*

In *Catoptricæ* parte *Directa Theorematum* horum usus est multiplex. Nam ex iis per communem Analysin innotescet an Curva quævis proposita pro Radiis parallelis, convergentibus vel divergentibus *Focum* habeat: Et, si habere constet, an is *Geometricus* sit, vel tantum *Physicus*: Et, si *Focum Geometricum* habere constiterit, an absolutè vel cum quibusdam conditionibus talis competat *Focus* Curvæ propositæ. Deniq; per hæc *Theoremata* ipsius *Foci* tam *Physici* quam *Geometrici* a Curvæ vertice *B* distantia invenietur.

1. Ope *Æquationis* Curvam propositam *BNR* definientis exterminentur *x* & *p* ex *Theoremate*, quod casui proposito convenit; & tum inveniatur valor Analyticus quantitatis *z* per *y* & determinatas expressus.

2. Si in prædicto valore quantitatis *z* non occurrat *y*, tum Curvæ propositæ competit *Focus Geometricus* absolutè, cujus a puncto dato *B* distantiam valor iste quantitatis *z* determinat.

3. Sed si *y* in illo quantitatis *z* valore occurrat, tum omnes termini, in quibus *y* est ejusdem dimensionis, ponantur nihilo æquales, & ex reductione harum *Æquationum* invenientur conditiones necessariæ ut *y* evanescat ex valore prædicto, adeoq; ut Curva proposita, in Casu istius *Theorematis*, *Focum* habeat *Geometricum*, cujus a puncto dato *B* distantiam valor iste (rejectis omnibus terminis in quibus *y* occurrit) determinat.

4. Si *Æquationum*, quæ per *Reg.* 3. inveniuntur, reductiones aliquid absurdi involvant, tum Curva proposita *Focum* non habet *Geometricum*.

5. Si

5. Si y occurrat in omnibus terminis valorem quantitatis z exprimentibus, tum nec *Geometricus* nec *Physicus Focus* Curvæ competit propositæ.

6. Si in valore quantitatis z unus sit (pluresve) terminus determinatus, tum Curva proposita *Focum* habet *Physicum*, quamvis per *Reg.* 4. non habere *Geometricum* innotescat: Et *Foci* hujus *Physici* a puncto dato B distantiam valor quantitatis z (rejectis terminis in quibus y occurrit) determinat: Sed ut salvo calculo rejici possunt termini, quos y ingreditur, oportet ut Abscissa y sit exigua, *i. e.* ut Curvæ propositæ pars parva BN capiatur.

SCHOLIUM.

Sæpissimè Calculum præcedentem abbreviare licet in hunc modum. Per *Æquationem* quæ Curvam BNR definit, exterminentur quantitates x & p ex debito *Theoremate*; quod postea a surdis & fractis (si quæ occurrant) liberetur. Tum comparentur termini *Theorematis* sic reducti juxta cognitas Comparationum Leges; *i. e.* comparando terminos, in quibus y eandem obtinet compositionem. Ex his comparationibus tot orientur *Æquationes*, quot in *Theoremate* reducto diversas habet y dimensiones. Ex harum *Æquationum* reductione invenietur valor quantitatis z determinatus, adeoq; *Foci Geometrici* K a puncto dato B distantia, nec non invenientur *Foci Geometrici* conditiones: Vel (si Curva proposita talem non habeat) *Foci Geometrici* impossibilitas detegetur.

EXEMPLUM I.

Sit BNR (*Fig.* 1. *Prop.* 1.) Parabola ad cujus axem BM incidentes MN sint paralleli.

Sit r latus rectum Parabolæ, unde $ry = x^2$, & $p = \frac{1}{2}r$, ideoq; per *Reg.* 1, in *Theor.* 1. pro x^2 substitute ry, & $\frac{1}{2}r$ pro p, erit $\frac{1}{4}rr + ry - rz = ry$; unde $z = \frac{1}{4}r = BK$: Unde habetur *Focus Geometricus* K; & quia y non occurrit in hoc quantitatis z valore, ideo per *Reg.* 2. pro Radiis incidentibus ad axem parallelis Parabola absolutè habet *Focum Geometricum*.

EXEM-

EXEMPLUM II.

Sit BNR (*Fig.* 1, 2. *Prop.* 1.) arcus Circuli, cui centrum C, fintq; Radii incidentes MN ad axem BCM paralleli.

Esto Circuli Radius $BC = n$; unde $2ny - y^2 = x^2$, & $p = n - y$. Hi valores quantitatum x^2 & p in *Theor.*.1. substituti dant $n^2 - 2nz + 2zy = 2ny$; quæ reducta dat $z = \frac{n^2 - 2ny}{2n - 2y}$. Jam ut *Focus* sit *Geometricus*, oportet ut valor quantitatis z sit determinatus; unde (per *Reg.* 3.) $-2ny = 0$, & $-2y = 0$: Et quia hæ *Æquationes* absurdum involvunt, ideo (per *Reg.* 4.) Circulus in hoc casu non habet *Focum Geometricum*; quare (per *Reg.* 6.) rejectis $2ny$, & $2y$ erit $z = \frac{n^2}{2n} = \frac{1}{2}n = BK$; i. e. *Fous Physicus* K est in puncto Radium CB bisecante.

EXEMPLUM III.

Incidant Radii divergentes in partes concavas (*Fig.* 1. *Prop.* 2.) vel convergentes in convexas (*Fig.* 2. *Prop.* 2.) partes Arcus circularis BNR, cujus centrum C, & Radius CB; qui productus transeat per punctum radians M vel punctum convergentiæ m.

Si in *Theor.* 2, pro x^2 & p substituantur $2ny - y^2$ & $n - y$, erit

$$\left.\begin{array}{ll} +n^2z^2 & +2n^2z \\ -2anz^2 & -2a^2z \\ & +2a^2n \end{array}\right\} y = \left.\begin{array}{ll} +a^2n^2 & +2nz^2 \\ -2a^2nz & -2az^2 \\ & +2an^2 \end{array}\right\} y.$$

Si juxta *Scholium* præcedens comparentur hujus *Æquationis* termini erit $n^2z^2 - 2anz^2 = a^2n^2 - 2a^2nz$; unde $z = \frac{an}{2a - n}$, & secunda comparatio erit $2n^2z - 2a^2z + 2a^2n. = 2nz^2 - 2az^2 + 2ann$: Unde $z = n$. Et proinde $n = \frac{an}{2a - n}$; unde $a = n$. Ex qua patet, Circulum in hoc casu tum tantùm habere *Focum Geometricum*, quando centrum C est punctum radians. Sed si

per *Reg.* 6.) rejiciantur omnes termini quos y ingreditur, ex reliquis invenietur $z = \frac{an}{2a - n} = BK$, quæ *Focum Physicum* K determinat. *Exemplum* itaq; propositum tres casus admittit.

CASUS I.

Quando $2a - n$ est quantitas affirmativa, *i. e.* quando est $2a \sqsubset n$, seu $a \sqsubset \frac{1}{2} n$, tum valor quantitatis z est affirmativus, adeoq; *Focus* K ad easdem partes jacebit respectu puncti *B*, ad quas jacet in *Fig.* 1 & 2. *Prop.* 2.

CASUS II.

Si $2a - n$ sit quantitas negativa, *i. e.* Si $2a \sqsupset n$ seu $a \sqsupset \frac{1}{2} n$, tum valor quantitatis z erit negativus, adeoq; *Focus* K jacebit ad partes contrarias (respectu puncti *B*) iis, quæ in *Fig.* 1 & 2. *Prop.* 2.

CASUS III.

Si $2a - n = 0$, seu $a = \frac{1}{2} n$, i. e. si punctum radians *M* vel punctum convergentiæ *m*, sit in puncto semidiametrum *BC* bisecante, tum valor quantitatis z est infinitus, adeoq; *Focus* reflexorum *K* est a puncto *B* infinitè distans, *i. e.* reflexi sunt paralleli.

EXEMPLUM IV.

Sit *BNR* arcus circuli, & Radii incidentes sint ad concavam convergentes versus *m* (*Fig.* 1. *Prop.* 3.) vel a puncto radiante *M* ad convexam divergentes ut in *Fig.* 2. *Prop.* 3. sitq; circuli Radius $BC = n$.

Ex *Theor.* 3. ponendo $2ny - y^2$ & $n - y$ pro x^2 & p invenietur $\overline{n^2 + 2an} \times z^2 + \overline{2n^2z - 2a^2z + 2a^2n + 2an^2} \times y = n^2 - 2nz \times a^2 + \overline{2n + 2a} \times z^2 y$. Ex qua, rejectis terminis quos y ingreditur, invenies $z = \frac{an}{2a + n} = BK$.

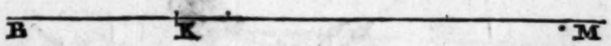
Usus

Usus Theorematum præcedentium in parte Catoptricæ inversa.

Ex Calculi fluxionum principiis $p = \frac{x\dot{x}}{\dot{y}}$, ideo si in debito *Theoremate* (ex *Problematis* conditione facillimè eligendo) substituatur hic valor subnormalis p, habebitur *Æquatio* fluxionalis; cujus fluens est invenienda per Methodos receptas vel in Libro primo de Calculo fluentium jam expositas; hæc fluens exprimet relationem inter y & x, seu definiet naturam Curvæ quæsitæ, quæ Luminis Radios juxta *Problematis* conditionem reflectet.

EXEMPLUM.

Invenire Curvam, quæ radios parallelos excipiens efficiet ut reflexi in dato puncto colligantur.



Vertex Curvæ quæsitæ sit in puncto dato vel assumpto B; sitq; K punctum datum, in quo incidentium parallelorum Reflexi sunt colligendi. Sit BM recta per puncta data B, K transiens. Jam quia *Problema* propositum pertinet ad casum *Theorematis* 1, ideo in *Theor.* 1, pro p substituatur $\frac{x\dot{x}}{\dot{y}}$, & post debitam reductionem invenietur.

$$x\dot{x}^2 + 2y\dot{x}\dot{y} - 2z\dot{x}\dot{y} = x\dot{y}^2.$$

Ut inveniatur hujus fluens assumatur $y = Ax + Bx^2$ &c. per quam (exterminatis y & $\dot{y}$) invenietur

$$\left.\begin{matrix} -2Az + 2A^2 \\ -4Bz \\ +1 \end{matrix}\right\} x = A^2 x \text{ \&c.}$$

Ut determinentur A, B erit prima comparatio $-2Az = 0$. Secunda $2A^2 - 4Bz + 1 = A^2$; unde $A = 0$, $B = \frac{1}{4z}$; unde $y = \frac{x^2}{4z}$,

seu

seu $4zy = x^2$; quæ est notissima Parabolæ proprietas, cujus latus rectum est $4z = 4BK$. Q. E. I.

SCHOLIUM.

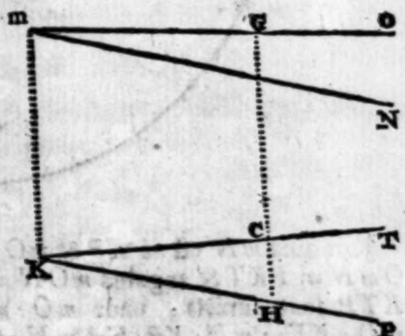
Per similem Calculum invenies relationem inter y & x, seu naturam Curvæ quæ radios Luminis secundum *Problematis* conditiones reflectet. Et proinde in his paucis omnia exhibentur quæ ad *Catoptricam generalem* intelligendam, necessaria videbantur.

DE DIOPTRICA

LIBER SECUNDUS.

LEMMA I.

SI a puncto *m* ducantur duæ quævis rectæ *m O*, *m N*, & ab alio quovis puncto *K* ducantur *KP* ad *m N* & *KT* ad *mO* parallelæ, erit angulus *O m N* angulo *PKT* æqualis. Ducatur recta *m K* & huic parallela quævis *G H* secans *m O*, *KT*, *KP* in punctis, *G*, *C*, *H*. Jam *KCG* = *OmK* (ob parallelismum linearum *m K*, *G H*) i. e. *KCG* = *OmN* + *NmK*: Sed

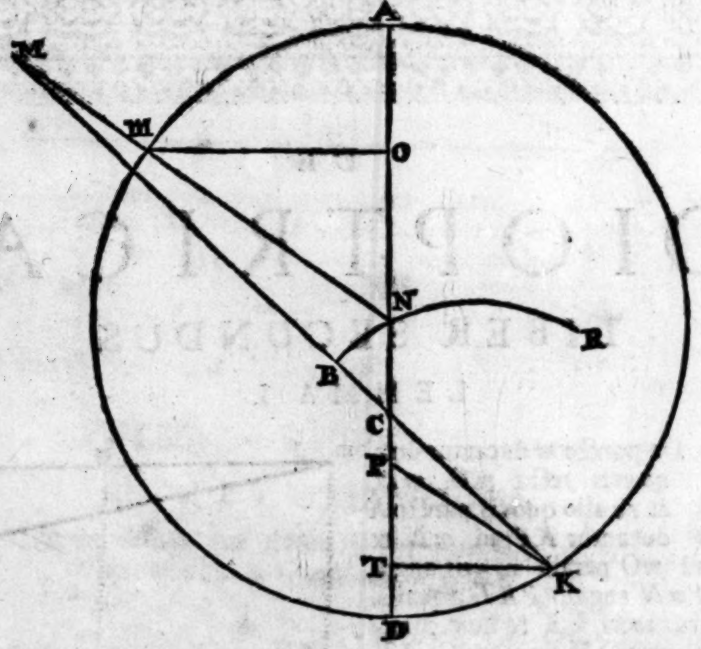


(ob parallelas *Nm*, *KH*) *NmK* = *KHC*; ergo erit etiam *KCG* = *OmN* + *KHC* : Sed *KCG* = *CKH* + *KHC* per 15. 1. *Eucl.* Ergo *CKH* + *KHC* = *OmN* + *KHC*; ergo *CKH* = *OmN*. Q. E. D.

LEMMA II.

Sit *M* punctum radians, a quo in Curvam quamvis *BNR* incidat luminis Radius *MN*, cujus refractus sit *NK*; sitq; sinus Anguli incidentiæ ad sinum Anguli Refractionis ut *s* ad *r*; esto *DN* normalis ad Curvam *BNR* in puncto incidentiæ *N*; secet quæ recta *MK* normalem in *C*, erit *r* × *CM* × *KN* = *S* × *CK* × *MN*.

A puncto K ducatur KP ad MN parallela & secans normalem in P. Centro N & radio NK describatur Circulus secans incidentem in m & Curvæ normalem in A, D; a punctis m, K demitte mO, KT in ACB perpendiculares.



Jam quia mN est ad KP & mO ad KT parallela; ideo per *Lem.* 1. $OmN = PKT$ & angulus $mON = KTP$; ergo triangula illa mON, KTP sunt similia; unde $mO : mN :: KT : KP$. Et alternando $mO : KT :: mN : KP$: Sed $mN = KN$; ergo $mO : KT :: KN : KP$. Sed ex hypothesi $mO : KT :: s : r$; ergo $KN : KP :: s : r$: Sed in demonstratione *Lemmatis* lib. 1. erat $KP = \frac{CK \times MN}{CM}$; ergo etiam $KN : \frac{CK \times MN}{CM} :: s : r$. Quæ Analogia in *Æquationem* mutata dat $r \times KN \times CM = s \times CK \times MN$. Q. E. D.

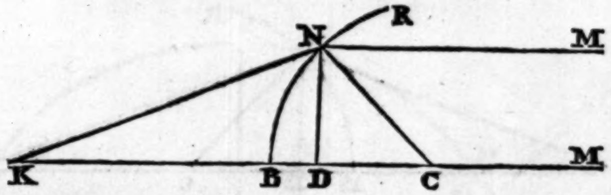
COROLLARIUM.

Quando Radii incidentes sunt paralleli, tum erit $r \times KN = s \times CK$; nam in hoc casu $CM = MN =$ infinito.

PRO-

PROPOSITIO I.

In qua traditur Theorema generale *pro Radiis parallelis in superficiem concavam incidentibus.*

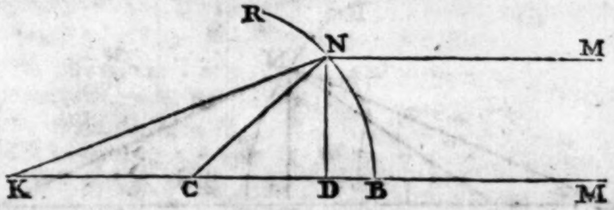


Sit BNR Curva quævis cujus axis BM, huic parallelus sit Radius incidens MN, cujus refractus sit NK axem BM secans in puncto K. Sintq; ND ad BM, & NC ad Curvam BNR normales; designentur autem quantitates ut in *Catoptrica* scil. $BD = y$, $DN = x$, $DC = p$, $BK = z$: Unde $KN = \sqrt{\overline{z+y}|^2 + x^2}$, $CK = p + y + z$; ergo $r\sqrt{\overline{z+y}|^2 + x^2} = s \times \overline{z+y+p}$; per *Coroll. Lemma* 2. hæc *Æquatio*, dat

$$r^2 \times \overline{z^2 + 2zy + y^2 + x^2} = s^2 \times \overline{z+y+p}|^2$$

PROPOSITIO II.

In qua traditur Theorema generale *pro Radiis parallelis in superficiem convexam incidentibus.*



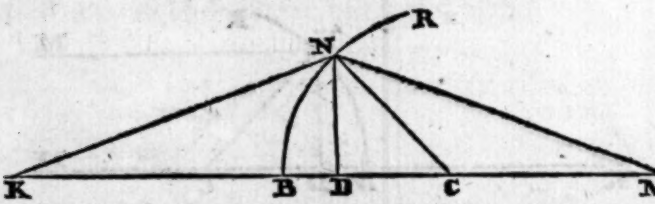
In Curvam quamlibet BNR incidat Radius MN ad Curvæ axem BD parallelus, cujus refractus sit NK axem secans in puncto K; designentur quantitates ut prius scil. $BD = y$, $DN = x$, $DC = p$, $BK = z$: Unde $KN = \sqrt{\overline{z-y}|^2 + x^2}$; $CK = z - y - p$; ergo $r\sqrt{\overline{z-y}|^2 + x^2} = s \times \overline{z-y-p}$ per *Coroll. Lem.* 2. hæc a signo radicali liberata dat hoc *Theorema*

$$r^2 \times \overline{z^2 - 2zy + y^2 + x^2} = s^2 \times \overline{z-y-p}|^2.$$

PRO-

PROPOSITIO III.

In qua traditur Theorema generale *pro Radiis divergentibus in superficiem concavam incidentibus.*

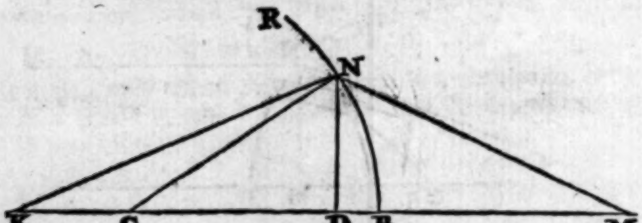


Sit BNR Curva quævis, ejus axis BM, punctum radians M, Radius incidens MN, refractus NK axem secans in K. Sitq; $BM = a$, cæteris (ut prius designatis, erit $KN = \sqrt{\overline{z+y}|^2 + x^2}$, $CM = a - y - p$, $CK = z + y + p$, $MN = \sqrt{\overline{a-y}|^2 + x^2}$; unde per *Lem.* 2. invenietur $r \times \overline{a - y - p} \times \sqrt{\overline{z+y}|^2 + x^2} = s \times \overline{z+y+p} \times \sqrt{\overline{a-y}|^2 + x^2}$ quæ a surdis liberata est

$$r^2 \times \overline{a-y-p}|^2 \times \overline{z^2+2zy+y^2+x^2} = s^2 \times \overline{z+y+p}|^2 \times \overline{a^2-2ay+y^2+x^2}.$$

PROPOSITIO IV.

In qua traditur Theorema generale *pro Radiis divergentibus in superficiem convexam incidentibus.*



Omnibus ut in præcedenti designatis, erit

$$KN = \sqrt{\overline{z-y}|^2 + x^2},\ CM = a + y + p.$$
$$MN = \sqrt{\overline{a+y}|^2 + x^2},\ CK = z - y - p.$$

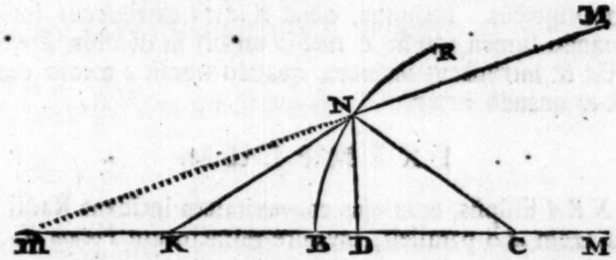
Unde per *Lem.* 2. habebitur *Theor.* quæsitum $r \times \overline{a+y+p}\sqrt{\overline{z-y}|^2 + x^2} = s \times \overline{z-y-p}\sqrt{\overline{a+y}|^2 + x^2}$, quod a surdis liberatum est

$$r^2 \times \overline{a+y+p}|^2 \times \overline{z^2-2zy+y^2+x^2} = s^2 \times \overline{z-y-p}|^2 \times \overline{a^2+2ay+y^2+x^2}.$$

PRO-

PROPOSITIO V.

In qua traditur Theorema generale *pro Radiis convergentibus qui incidunt in superficiem concavam.*



Sit m punctum Convergentiæ ad quod tendunt omnes Radii incidentes MN; tum compositis ut in præcedenti, fit $Bm = a$, erit

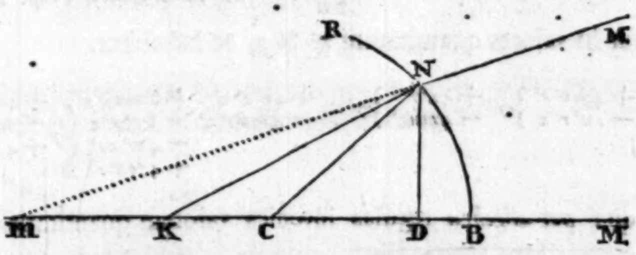
$$KN = \sqrt{\overline{z+y}|^2 + x^2},\ Cm = a + y + p.$$
$$Nm = \sqrt{\overline{a+y}|^2 + x^2},\ CK = z + y + p.$$

Unde per *Lemma* 2. habebitur *Theorema* quod a surdis liberatum erit

$$r^2 \times \overline{a+y+p}|^2 \times \overline{z^2+2zy+y^2+x^2} = s^2 \times \overline{z+y+p}|^2 \times \overline{a^2+2ay+y^2+x^2}.$$

PROPOSITIO. VI.

In qua traditur Theorema generale *pro Radiis convergentibus qui incidunt in superficiem convexam.*



Omnibus designatis ut in præcedenti invenietur

$$KN = \sqrt{\overline{z-y}|^2 + x^2},\ Cm = a - y - p,$$
$$Nm = \sqrt{\overline{a-y}|^2 + x^2},\ CK = z - y - p.$$

Unde per *Lem.* 2. habetur *Theor.* quæsitum, quod a surdis liberatum est

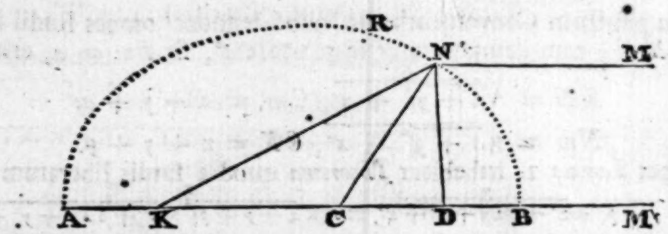
$$r^2 \times \overline{a-y-p}|^2 \times \overline{z^2-2zy+y^2+x^2} = s^2 \times \overline{z-y-p}|^2 \times \overline{a^2-2ay+y^2+x^2}.$$

 Notan-

Notandum. Quod in horum *Theorematum* Uſu adhibendæ ſint Regulæ, quas in *Catoptrica* tradidi. Quando valor quantitatis z eſt affirmativus, tum *Focus* K erit realis, ſeu Radii refracti in puncto K colligentur: Sin valor quantitatis z ſit negativus, tum *Focus* K erit punctum Diſperſûs. Denique, quod Radius extrinſecus incidere dicitur, quando lumen tranſit è medio rariori in denſius, *i. e.* quando $s \sqsubset r$: Ut & intrinſecus incidere, quando tranſit *e* medio denſiori in rarius, *i. e.* quando $s \sqsupset r$.

EXEMPLUM.

Sit $BNRA$ Ellipſis, & in ejus convexitatem incidant Radii MN ad Ellipſeos axem AB paralleli, invenire refractorum *Focum* K.



Sit Axis tranſverſus $AB = n$, latus rectum m; Abſciſſa $BD = y$, Ordinata $DN = x$, ſubnormalis $DC = p$, jam ex natura Ellipſeos $mny - my^2 = nx^2$, $p = \dfrac{mn - 2my}{2n}$: In *Theoremate Prop.* 2. ſubſtituantur hi valores quantitatum x^2 & p, & habebitur.

$$4n^2r^2z^2 \left.\begin{matrix} \pm 4mn^2r^2 \\ - sn^2r^2z \end{matrix}\right\} y \left.\begin{matrix} \pm 4n^2r^2 \\ - 4mnr^2 \end{matrix}\right\} y^2 = \begin{matrix} \pm 4n^2s^2z^2 \\ - 4mns^2z^2 \end{matrix} \left.\begin{matrix} + 8mns^2z \\ - 8n^2s^2z \\ - 4m^2ns^2 \\ + 4mn^2s^2 \end{matrix}\right\} y \left.\begin{matrix} \pm 4m^2s^2 \\ - 8mns^2 \\ + 4n^2s^2 \end{matrix}\right\} y^2$$

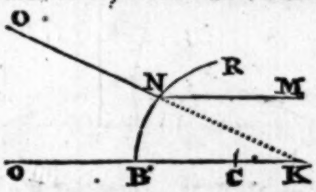
Ex qua per *Algebræ* regulas invenies valorem quantitatis z, ſcil. $z = \dfrac{\sqrt{r^2s^2m^2n^2 + Ay + By^2} + s^2mn + Cy}{2ns^2 - 2nr^2}$; Ubi compendii gratia ponitur $4mn^2r^2s - 4m^2nr^2s^2 - 4mn^2r^4 = A$, $4mnr^4 + 4m^2r^2s^2 - 4mnr^2s^2 = B$, $2s^2n - 2s^2m - 2r^2n = C$. Jam ut in hoc caſu Ellipſis habeat *Focum Geometricum*, oportet hic valor quantitatis z ſit determinatus; efficiendum itaq; ut y ex prædicto valore evaneſcat; ponendo $A = 0$, $B = 0$, $C = 0$; hæ *Æquationes*

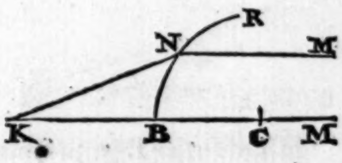
tiones (restitutis quantitatum *A*, *B*, *C* valoribus) reductæ dabunt $m = \frac{\overline{s^2 - r^2} \times n}{s^2}$; quæ ratio lateris transversi ad rectum efficiet ut y evanescat ex valore quantitatis z, & tum $z = \frac{\sqrt{r^2 s^2 m^2 n} + s^2 mn}{2n \times \overline{s^2 - r^2}}$ $= \frac{rsm + s^2 m}{2s^2 - 2r}$, in qua si pro m substituas ejus valorem modo inventum erit $z = \frac{\overline{r + s} \times n}{2s} = BK$.

Patet itaq; Ellipsin habere *Focum Geometricum* pro Radiis parallelis; non tamen absolute, sed cum hac limitatione, ut sit latus transversum n ad latus rectum m ut s^2 ad $s^2 - r^2$. Ex Conicis vero patet punctum K sic inventum esse Ellipseos Umbilicum remotiorem. Et hanc in luminis Refractione proprietatem Ellipsi competeret primus detexit Insignis *Cartesius*. Eodem modo eâdemq; facilitate Curvæ cujusvis propositæ *Focus* per præcedentia *Theoremata* invenietur. Cæteris autem missis, in Circuli *Foco* determinando *Theorematum* horum Usus illustrabitur; *Dioptrica* enim Sphærica ob rationes plures cæteris est preferenda. Utq; hæc sine fastidiosa verborum repetitione absolvatur, sit semper in sequentibus C centrum Circuli Sphæram refringentem generantis, B vertex ac Abscissæ y generantis, ejus Semidiameter $CB = m$, adeoq; $2my - y^2 = x^2$, $p = m - y$; & in unoquoq; *Problemate Fig.* 1. pertineat ad Radios extrinsecùs, *Fig.* 2. ad Radios intrinsecùs incidentes.

PROBLEMA I.

Data Superficie Sphærica concava, in quam incidant Radii paralleli MN, *Focum* K refractorum invenire.



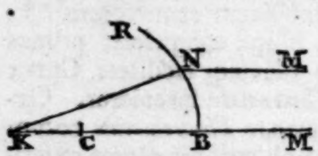


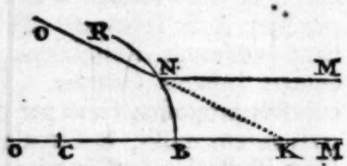
Quia *Problema* propositum pertinet ad *Theorema Prop.* 1. ideo substituantur $2my - y^2$ & $m - y$ pro x^2, & p in illo *Theoremate*, eritq; $r^2 \times \overline{z^2 + 2zy + 2my} = s^2 \times \overline{z + m}|^2$; si hæc tractetur juxta

juxta Regulas in *Catoptrica* traditas invenietur (per *Reg.* 4.) Circulum non habere *Focum Geometricum*: Rejiciantur itaq; termini (per *Reg.* 6.) in quibus y occurrit, eritq; $r^2z^2 = s^2 \times \overline{z + m}|^2$; unde $z = \frac{sm}{r - s}$ $= BK$. Ex quo patet, quod si fuerit $s \sqsubset r$ (ut ex hyp. in *Fig.* 1.) tum valor quantitatis z erit negativus, & proinde *Focus* K erit negativus, *i. e.* Superficies Sphærica refringens disperget Radios refractos NO, quasi a puncto K prodirent. Sin fuerit $s \sqsupset$ (ut in *Fig.* 2. ex hyp.) tum z erit valoris affirmativi, adeoq; refracti NK in puncto K colligentur.

PROBLEMA II.

Invenire *Focum* K, quando Radii paralleli MN incidunt in Superficiem Sphæricam convexam.

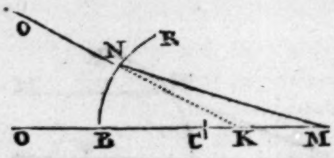


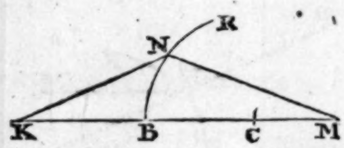


Substituendo $2my - y^2$, & $m - y$ pro x^2, & p in *Theoremate Prop.* 2. invenietur $z = \frac{sm}{s - r} = BK$. Patet itaque, quod si $s \sqsubset r$, tum *Focus* K erit punctum concursûs; Sin $s \sqsupset r$, tum K erit punctum dispersûs (ut in *Fig.* 2.)

PROBLEMA III.

Invenire *Focum* K, quando Radii divergentes MN incidunt in Superficiem Sphæricam concavam.





In *Theoremate Prop.* 3. substituantur $2my - y^2$, & $m - y$ pro x^2 & p, eritq;

$$\left.\begin{matrix} r^2b^2z^2 \mp 2r^2b^2z \\ + 2r^2b^2m \end{matrix}\right\} y = s^2 \times \overline{az + am}|^2 \left.\begin{matrix} - 2s^2bz^2 \\ - 4s^2bmz \\ - 2s^2bm^2 \end{matrix}\right\} y.$$

Ubi

Ubi compendii gratia posui $a - m = b$. Juxta *Scholium* Regulis in *Catoptrica* traditis subjunctum erit prima comparatio $r^2 b^2 z^2 = s^2 \times \overline{az + am}|^2$; unde $z = \frac{sam}{rb - sa} = \frac{sam}{ra - sa - rm}$; & secunda comparatio est terminorum in quibus y occurrit scil. $2r^2 b^2 \times \overline{z + m} = -2s^2 b \times \overline{z^2 + 2mz + m^2}$, quæ reducta dabit $z = \frac{r^2 m - s^2 m - r^2 a}{s^2}$. Jam ut Circulus in hoc casu *Focum* habeat *Geometricum*, oportet ut hi duo valores quantitatis z sint æquales, fiat ergo $\frac{r^2 m - s^2 m - r^2 a}{s^2} = \frac{sam}{ra - sa - rm}$; quæ post debitam reductionem dabit $a = \frac{s + r}{r} m$. Unde constat Circulum habere *Focum Geometricum* quando Radii divergentes incidunt in partes ejus concavas, dummodo ea sit puncti radiantis M a vertice B distantia, ut $BM = \frac{r + s}{r} \times CB$. Sed in omni alia puncti radiantis distantia *Focus* refractorum est tantùm *Physicus*, cujus a vertice B distantia ex comparatione prima determinatur scil. $z = \frac{sam}{ra - sa - rm} = BK$. Unde patet, quod, si Radii incidant extrinsecus (ut ex hyp. in *Fig.* 1.) *Focus* K sit negativus, *i. e.* K erit punctum dispersûs; nam in hoc casu erit $ra - sa - rm$ quantitas negativa: Sed si Radii incidant intrinsecùs, tum pro varia puncti radiantis M a vertice B distantia, *Focus* interdum erit affirmativus, interdum negativus, & in uno casu Radios refractos reddet parallelos: Quæ omnia ex modò invento quantitatis z valore per prima *Algebræ* principia facilè deduci possunt.

CASUS I.

Si *Focus* sit *Geometricus*, *i. e.* si $a = \frac{sm + rm}{r}$; tum K erit punctum dispersûs. Nam si in valore quantitatis z substituatur $\frac{sm + rm}{r}$ pro a, erit $z = \frac{sm + rm}{-s} = BK$.

CASUS II.

Si $ra - sa - rm = 0$, i. e. si $a = \frac{rm}{r-s}$, tum $z = BK =$ infinito, *i. e.* refracti NK erunt ad axem BC paralleli.

CASUS III.

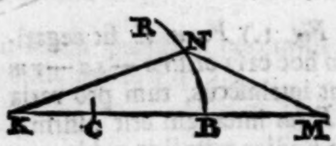
Si $ar \sqsubset sa + rm$, i. e. si $a \sqsubset \frac{rm}{r-s}$, tum *Focus Physicus* K erit affirmativus, *i. e.* K erit punctum concursûs.

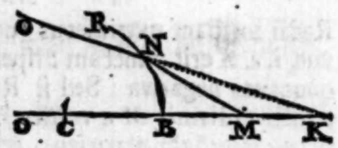
CASUS IV.

Si $ra \sqsupset sa + rm$, i. e. si $a \sqsupset \frac{rm}{r-s}$, tum *Focus* erit negativus, seu K erit punctum dispersûs.

PROBLEMA IV.

Invenire *Focum K*, quando Radii divergentes incidunt in Superficiem Sphæricam convexam.





Si in *Theoremate Propositionis* 4. pro x^2 & p substituantur $2my - y^2$ & compendii gratia ponatur $b = a + m$, erit

$$\left.\begin{array}{r} r^2b^2z^2 + 2r^2b^2m \\ - 2r^2b^2z \end{array}\right\} y = s^2 \times \overline{az - am}|^2 \left.\begin{array}{l} + 2s^2bz^2 \\ - 4s^2bmz \\ + 2s^2bm^2 \end{array}\right\} y.$$

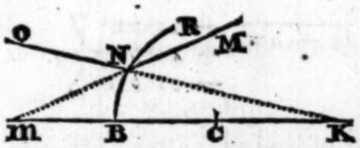
Si hæc *Æquatio* tractetur eodem modo, quo præcedens, invenietur Circulum in hoc casu non habere *Focum Geometricum*, ideoq; rejectis terminis, quos y ingreditur erit $\overline{rbz}|^2 = s^2 \times \overline{az - am}|^2$: Unde $z = \frac{sam}{sa - ra - rm} = BK.$

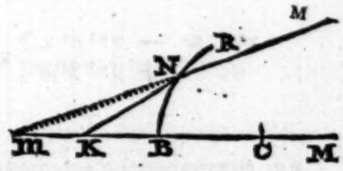
Patet

Patet itaq; quod pro varia magnitudine quantitatis $a = BM$, i. e. pro varia distantia puncti radiantis M a vertice B, erit $sa - ra - rm$ major, minor aut æqualis nihilo; in primo Casu z erit quantitas affirmativa adeoq; K erit punctum concursûs; in secundo erit negativæ, adeoq; K erit punctum dispersûs; in tertio z erit infinita, adeoq; Radii erunt paralleli.

PROBLEMA V.

Invenire *Focum K*, quando Radii Convergentes incidunt in concavam Superficiem Sphæricam.





Sit m radiorum incidentium punctum convergentiæ, adeoq; & $Bm = a$; si in *Theoremate Prop.* 5. pro x^2 & p substituantur earum valores $2my - y^2$, & $m - y$; & ponatur $b = a + m$, erit

$$\left.\begin{array}{l} r^2b^2z^2 + 2r^2b^2z \\ \qquad\quad + 2r^2b^2m \end{array}\right\} y = s^2 \times \overline{az + am}|^2 \left.\begin{array}{l} + 2s^2bz^2 \\ + 4s^2bmz \\ + 2s^2bm^2 \end{array}\right\} y.$$

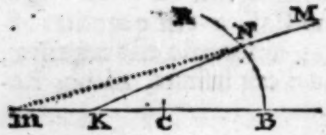
Quæ si tractetur, ut *Æquatio Prob.* 4. invenietur $ra + rm + sm = 0$, adeoq; in hoc casu Circulus non habet *Focum Geometricum*, rejectis itaq; terminis quos y ingreditur, erit $r^2b^2z^2 = s^2 \times \overline{az + am}|^2$; unde $z = \dfrac{sam}{ra - sa + rm} = BK$.

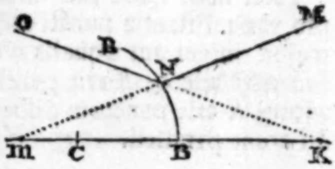
Ex qua patet, quod pro varia quantitatis a magnitudine, $ra - sa + rm$ fit major, minor vel æqualis nihilo; & proinde valor quantitatis z in primo casu erit affirmativus, adeoq; K erit Radiorum refractorum punctum concursûs; in secundo erit negativus, adeoq; K erit refractorum punctum dispersûs; & in tertio casu erit infinitus, adeoq; Radii refracti erunt ad axem BC paralleli.

PROBLEMA VI.

Invenire *Focum K*, quando Radii Convergentes MN incidunt in Superficiem Sphæricam convexam.

Sit





Sit m punctum Convergentiæ, $Bm = a$. Jam si in *Theoremate Prop.* 6. pro x^2 & p substituantur $2my - y^2$, $m - y$, & compendii causa ponatur $b = a - m$, erit.

$$\left.\begin{matrix} r^2 b^2 z^2 - 2r^2 b^2 z \\ + 2r^2 b^2 m \end{matrix}\right\} y = s^2 \times \overline{az - am}|^2 \left.\begin{matrix} - 2s^2 bz^2 \\ + 4s^2 bmz \\ - 2s^2 bm^2 \end{matrix}\right\} y.$$

Ex comparatione terminorum, in quibus deest y, erit $\overline{rbz}|^2 = s^2 \times \overline{az - am}|^2$; unde $z = \frac{sam}{sa - rb}$: Et ex comparatiône terminorum in quibus y occurrit, erit $-2r^2b^2z + 2r^2b^2m = -2s^2bz^2 + 4s^2bmz - 2s^2bm^2$; unde $z = \frac{r^2b + s^2m}{s^2}$: Et proinde $\frac{r^2b + s^2m}{s^2} = \frac{sam}{sa - rb}$; quæ reducta dabit $rsa - r^2a + r^2m - s^2m = 0$: Hæc divisa per $s - r$ dat $ra - rm - sm = 0$; unde $a = \frac{rm + sm}{r}$. Unde patet, quod si talis fuerit puncti Convergentiæ m a vertice B distantia, tum in hoc casu *Focus* inventus K erit *Geometricus*.

Sed in omni alia puncti m distantia *Focus* tantum *Physicus* est ex prima comparatione determinatus scil.

$$z = \frac{sam}{sa - ra + rm} = BK.$$

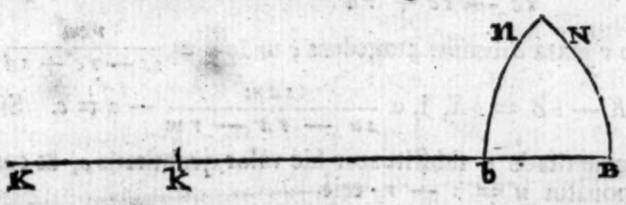
Ex qua colligitur $sa - ra + rm \sqsubset 0$, vel $sa - ra + rm \sqsupset 0$, vel $sa - ra + rm = 0$, adeoq; Radios colligi in puncto K, vel a puncto K dispergi, vel ad axem BC parallelos evadere; pro varia puncti m a puncto B distantia.

Problema

Problema generale I.

Lentis cujusvis datæ Focum *invenire.*

Sint BN, bn duæ lineæ quæcunque, ex quarum rotatione circa rectam Bb generatur data lens $BNnb$. Sitq; BN linea (recta vel Curva) in quam extrinsecus incidunt Radii paralleli, a puncto dato divergentes, vel ad punctum datum convergentes.



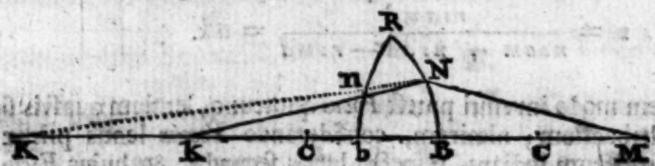
Per *Theoremata* præcedentia inveniatur *Focus* K Radiorum refractorum, quorum incidentes in Curva (vel recta) BN refractionem patiuntur. Tum habeatur *Focus* sic inventus K, punctum Convergentiæ ad quod Radii in alteram Curvam bn intrinsecus incidant, tum ex datis naturâ lineæ bn, puncto Convergentiæ K, ejusq; a vertice b distantiâ, inveniatur refractorum *Focus* k per *Theoremata generalia* præcedentia; eritq; *Focus* sic inventus k lentis datæ *Focus* quæsitus. *Q.E.I.*

SCHOLIUM.

Quia $\frac{s}{r}$ denotat rationem Sinus incidentiæ ad Sinum prioris refractionis in BN factæ, ideo $\frac{r}{s}$ est ratio sinus incidentiæ ad Sinum posterioris refractionis factæ in bn; ideo, quando per *Theoremata generalia* investigatur secundæ refractionis *Focus*. k, scribatur in debito *Theoremate* r pro s, & è contra s pro r; nam Radii in BN incidunt extrinsecùs; & in bn semper intrinsecùs incidere habentur.

EXEMPLUM.

Lentis cujusvis Sphæricæ Focum *invenire.*



Sit BRb lens Sphærica Convexo-convexa, M punctum radians, $BM = a$; arcus BR Radius $CB = m$, K *Focus* primæ refractionis a

BNR factæ; unde (per *Prob.* 4) $z = \frac{sam}{sa - ra - rm} = BK$. Itaq; refracti *Nn* intrinsecus in Superficiem concavam *bnR* incidentes ad punctum *K* convergunt; sit ergo $bK = c$, $bB = e$, arcus *bnR* Radius $cb = d$; *Focus k*, ejus a vertice *b* distantia $bK = u$; ideo (per *Prob.* 5.) $u = \frac{scd}{rc - sc + rd} = bk$: Sed scribendum est *r* pro *s*, & *s* pro *r* juxta *Scholium* præcedens; unde $u = \frac{rcd}{sc - rc + sd} = bk$: Sed $BK - bB = bK$, i. e. $\frac{sam}{sa - ra - rm} - e = c$. Si itaq; in valore quantitatis *u* substituatur hic valor quantitatis *c*, & compendii causa ponatur $n = s - r$, erit

$$u = \frac{rsamd - nraed + rremd}{nsam - n^2ae + nrme + nsad - rsmd} = bk.$$

1. Quando punctum radians *M* est infinite distans, *i. e.* quando Radii incidentes in *BNR* sunt paralleli, tum a = infinito. 2. Si *BNR* sit linea recta, *i. e.* si lens sit plano-convexa, tum m = infinito: 3. Si *bnR* sit linea recta, *i. e.* si lens sit convexo plana, tum d = infinito. 4. Si *BNR* sit versus *M* cava, *i. e.* si lens sit concavo-convexa, tum pro $+ m$ scribatur $- m$, & pro $- m$ scribatur $+ m$: Et similiter, si *bnR* sit versus *M* cava, scribatur $- d$ pro $+ d$, & $+ d$ pro $- d$. 5. Denique, si Radii incidentes fuerint convergentes, pro $+ a$ scribatur $- a$, & pro $- a$ scribatur $+ a$; & in hoc casu (a) denotat distantiam inter verticem *B* & punctum Convergentiæ.

COROLLARIUM.

Quoniam plerumq; negligi potest Lentis craffities *Bb*, ideo rejiciantur omnes termini quos *e* ingreditur, & tum *Æquatio* fiat simplicior scil.

$$u = \frac{rsamd}{nsam + nsad - rsmd} = bk.$$

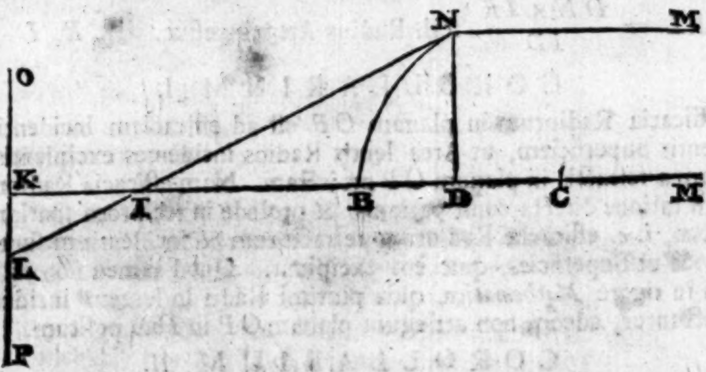
Eodem modo inveniri potest *Focus* quotcunq; lentium quasvis figuras datas habentium; nimirum, considerando *Focum* lentis primæ tanquam punctum radians, respectu lentis secundæ, & hujus *Focus* tanquam lentis tertiæ punctum radians; & sic porro.

Problema

Problema generale II.

Foci Physici aberrationem invenire.

Distantia inter *Focum Physicum K* & punctum *I*, in quo extremi incidentis *MN* refractus *NI* secat axem *BM*; scil. *KI* vocatur ab *Hugenio Aberratio*, quam invenire oportet.



Quoniam hæc aberratio inde oritur, quod ex valore quantitatis z rejecti fuerint termini, quos y ingreditur; unde imperfectus ejus valor dat BK: Et quoniam verus ejus valor dat BI, scil. quando ex *Theoremate* non rejiciuntur termini quos y ingreditur. Ideo horum valorum (veri scil. & imperfecti) differentia dabit Aberrationem quæsitam: Unde pro Curva quæcunq; BN habetur aberratio KI per *Theoremata* præcedentia.

EXEMPLUM.

Sit BN arcus Circuli, cujus centrum C, & Radius $CB = m$: Unde (per *Prob.* 1.) $z = \frac{sm}{r-s} = BK$, quando y rejiciatur ex *Theoremate Prop.* 1. Sed si y non rejiciatur, tum hoc *Theorema* dabit verum valorem quantitatis $z = \frac{r\sqrt{s^2m^2 - 2r^2my + r^2y^2} + s^2m - r^2y}{r^2 - s^2} = BI$; Unde horum differentia dabit aberrationem quæsitam scil. $\frac{sm}{r-s}$ $- \frac{r\sqrt{s^2m^2 - 2r^2my + r^2y^2} + r^2y - s^2m}{r^2 - s^2} = BK - BI = KI.$

Problema

Problema generale III.

Invenire Aream quæ ab omnibus refractis projicitur in planum ad axem normale & in Foco *positum.*

Cæteris positis ut in præcedenti, sit OP planum positum in K ad MBK normale; producatur refractus NI donec plano OP occurrat in L. Jam ob triangula similia IDN, IKL erit $ID : DN :: IK : KL$, seu $KL = \frac{DN \times IK}{ID}$, scil. Radius Areæ quæsitæ. *Q. E. I.*

COROLLARIUM I.

Efficacia Radiorum in planum OP est ad efficaciam incidentium in lentis Superficiem, ut Area lentis Radios incidentes excipientis ad Aream a refractis in planum OP projectam. Nam efficacia Radiorum est in ratione directa constipationis, & proinde in reciproca spatiorum ratione, *i. e.* efficaciæ Radiorum refractorum & incidentium sunt reciprocè ut Superficies, quæ eos excipiunt. Quod tamen non est verum in rigore *Mathematico*, quia plurimi Radii in lentem incidentes reflectuntur, adeoq; non attingunt planum OP in *Foco* positum.

COROLLARIUM II.

Ex hisce duæ quælibet lentes facile comparari, & utra earum Radios refractos melius colligat determinari potest. Nam illa lens melius colligit Radios seu majorem habet efficaciam, quæ æqualem Radiorum numerum in minorem (supra planum OP) aream projicit.

Notandum. Præter Aberrationem prædictam, quæ oritur a figura superficiei lumen inflectentis; alteram notabilem detexit Illustriss. *Newtonus*, ex qua pulcherrimam suam de Coloribus Theoriam deduxit, integroq; libro exposuit.

Usus Theorematum *præcedentium in parte* Dioptricæ *inversa.*

Datis *Foco* & puncto radiante Curvam refringentem invenies per Methodum in *lib.* 1. traditam; nimirum, substituatur $\frac{x\dot{x}}{\dot{y}}$ pro p in debito *Theoremate* lib. 2; & sic habebitur *Æquatio* fluxionalis; cujus fluens per Methodos supra expositas invenienda dabit relationem inter y & x, quæ definiet Curvæ quæsitæ Naturam. *Exempla* addere non erit opus, satis enim patet Methodus ex iis, quæ ad eandem illustrandam *Catoptrica* exhibentur.

FINIS.